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Exact diffusion coefficient of self-gravitating Brownian particles in two dimensions

P.H. Chavanis[a]

Laboratoire de Physique Théorique, Université Paul Sabatier, 118 route de Narbonne 31062 Toulouse, France

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Abstract. We derive the exact expression of the diffusion coefficient of a self-gravitating Brownian gas in two dimensions. Our formula generalizes the usual Einstein relation for a free Brownian motion to the context of two-dimensional gravity. We show the existence of a critical temperature T_c at which the diffusion coefficient vanishes. For $T < T_c$, the diffusion coefficient is negative and the gas undergoes gravitational collapse. This leads to the formation of a Dirac peak concentrating the whole mass in a finite time. We also stress that the critical temperature T_c is different from the collapse temperature T_* at which the partition function diverges. These quantities differ by a factor $1 - 1/N$ where N is the number of particles in the system. We provide clear evidence of this difference by explicitly solving the case $N = 2$. We also mention the analogy with the chemotactic aggregation of bacteria in biology, the formation of "atoms" in a two-dimensional (2D) plasma and the formation of dipoles or "supervortices" in 2D point vortex dynamics.

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1 Introduction

Recently, there was a renewed interest for systems with long-range interactions [1]. These systems have a very strange thermodynamics and exhibit peculiar features that are very different from those of more familiar systems with short-range interactions like neutral gases and plasmas. One striking property of systems with unshielded attractive long-range interactions is their ability to self-organize spontaneously into large-scale coherent structures. Some examples are provided by stars, globular clusters and galaxies in astrophysics, jets and vortices (e.g. Jupiter's great red spot, gulf stream,...) in two-dimensional geophysical flows, bacterial aggregates in biology, clusters in the Hamiltonian Mean Field (HMF) model etc. As a result of spatial inhomogeneity and non-extensivity, the ordinary thermodynamic limit ($N \to +\infty$ with N/V fixed) is clearly irrelevant for these systems and must be reconsidered [2,3]. On the other hand, the statistical ensembles are generically inequivalent and the choice of the relevant ensemble must be addressed specifically. Therefore, systems with long-range interactions are special and escape the ordinary rules of thermodynamics. Among all types of systems with long-range interactions, self-gravitating systems are probably the most fundamental [4]. These systems have a complex thermodynamics and present interesting phase transitions between "gaseous" and "clustered" states [5]. They can indeed undergo catastrophic collapse below a critical energy E_c in the microcanonical ensemble [6] or below a critical temperature T_c in the canonical ensemble [7] when gravitational attraction overcomes diffusive effects.

In astrophysics, the dynamics of self-gravitating systems is basically described by the Newton equations where the acceleration of a particle is equal to the gravitational force (per unit of mass) created by the other particles [8–10]. These equations have a Hamiltonian structure. For a large number of particles $N \gg 1$, we cannot follow the motion of each particle in detail and we must have recourse to statistical mechanics [4,5]. An isolated self-gravitating system is *conservative* (evolving at fixed energy E) so that the proper statistical ensemble is the microcanonical ensemble. This is the correct description of stellar systems such as globular clusters and galaxies. The Hamiltonian N-stars model has a very long history starting with Newton's *Principia Mathematica* in 1687.

In a recent series of papers, Chavanis and Sire [11–21] have introduced and systematically studied a model of self-gravitating Brownian particles. In this model, the particles interact gravitationally but they also experience a friction force and a stochastic force which mimick a coupling with a thermal bath of non gravitational origin. Therefore, the basic equations of motion consist in

[a] e-mail: chavanis@irsamc.ups-tlse.fr

a set of N coupled Langevin equations. This system is *dissipative* (evolving at fixed temperature T) so that the proper statistical ensemble to consider is the canonical ensemble. This Brownian model could find applications in the process of planetesimal formation in the solar nebula [22]. In this context, the dust particles experience a friction with the gas, a stochastic force due to turbulence and, when the dust layer becomes dense enough, self-gravity must be taken into account. On the other hand, some interesting analogies have been found between a self-gravitating Brownian gas and the process of chemotaxis in biology [17,23], the formation of large-scale vortices in 2D turbulence [24,25] and the Bose-Einstein condensation in the canonical ensemble [26]. At a more academic level, Brownian motion is a fundamental process in physics (pioneered by the works of Einstein and Smoluchowski) and it is clearly of interest to investigate the situation where N Brownian particles are coupled by a long-range potential of interaction, like the gravitational interaction. These various arguments may justify the study of the self-gravitating Brownian gas model.

Random walkers in interaction described by N coupled Langevin equations are also studied in soft matter physics to compute transport properties of systems consisting in many interacting particles such as supercooled liquids or the dynamics of colloids in solution [27,28]. In these examples, the potential of interaction is short-range and the system is spatially homogeneous at equilibrium. An interesting problem consists in determining the effective diffusion coefficient of a particle of the system. This is a very difficult problem and essentially perturbative approaches have been developed. For example, Dean and Lefèvre [29] consider a regime of weak coupling (small β with fixed $\beta\rho$) where diagrammatic expansions can be carried out. Interestingly, their approach suggests that the system will experience a glassy behaviour (identified by the divergence of the relaxation time and the vanishing of the diffusion coefficient) at some critical temperature T_c. However, since the expansions are valid at high temperatures, it is not clear whether their extrapolation to lower temperatures of order T_c is justified.

In this paper we show that, for the self-gravitating Brownian gas in two dimensions, the diffusion coefficient of a particle can be calculated *exactly* for any temperature. It is given by the following expression

$$D(T) = \frac{k_B T}{\xi m}\left(1 - \frac{T_c}{T}\right), \tag{1}$$

where T_c is the critical temperature

$$k_B T_c = (N-1)\frac{Gm^2}{4}. \tag{2}$$

For $T \gg T_c$, when self-gravity becomes negligible, we recover the Einstein relation [30]. In that case, the particles have a diffusive motion (corresponding to an evaporation of the system) slightly modified by self-gravity. For $T = T_c$, the diffusion coefficient vanishes and for $T < T_c$ it becomes negative implying finite time collapse. In that case, the system forms a Dirac peak containing the whole mass in a finite time. These different regimes have been studied in detail in the mean field approximation by analytically solving the Smoluchowski-Poisson system [12,19]. Here, we present complementary results that are valid beyond the mean field approximation. We stress that, contrary to the case of systems with short-range interactions like those evocated above [27–29], we calculate the diffusion coefficient of a particle in a gravitational system that is out-of-equilibrium and spatially inhomogeneous. The context is therefore very different from the case of soft matter physics [27–29]. We also stress that the result (1)-(2) only holds in two-dimensional gravity. Gravity is known to be critical in two dimensions because, dimensionally, the gravitational potential $u = Gm^2\ln(r_{ij})$ does not depend on the distance (the distance enters in a dimensionless logarithmic factor). This is the intrinsic reason why the Virial of the gravitational force and the diffusion coefficient can be calculated exactly in 2D gravity.

The paper is organized as follows. In Section 2, we develop a many-body theory of Brownian particles in interaction and derive exact kinetic equations valid for an arbitrary binary potential of interaction (Sect. 2.1). In Section 2.3, we derive the general expression of the Virial theorem for a Brownian gas with two-body interactions. When we consider the gravitational potential in two dimensions (Sect. 2.4), the Virial theorem takes a very simple form from which we can deduce the exact expression of the diffusion coefficient (Sect. 2.5). The mean field approximation is considered in Section 2.6. In Section 3, we obtain the exact equation of state of a self-gravitating gas in two dimensions. This equation of state, first derived in plasma physics for electric charges [31], is well-known. It is usually derived in the canonical ensemble from the partition function [4,32]. Here, we show that the same expression can be obtained in the microcanonical ensemble from the density of states. We also generalize its expression by allowing the particles to have different masses (or charges). In Section 4, we investigate the existence of statistical equilibrium for a two-dimensional self-gravitating system in the canonical ensemble. We show that the partition function exists only above a temperature T_*. This temperature differs from the critical temperature T_c appearing in the Virial theorem and in the equation of state by a factor $1 - 1/N$ where N is the number of particles in the system. We provide clear evidence of this difference by explicitly solving the case $N = 2$ in Section 5. This difference has been overlooked in the literature. However, this is essentially a curiosity because, in the thermodynamic limit $N \to +\infty$, the two temperatures coincide.

2 Many-body theory

2.1 Exact kinetic equations

In a space of dimension d, we consider a system of N Brownian particles confined within a domain (box) of volume V and interacting via a binary potential [2,3]. Their

dynamics is described by the coupled Langevin equations

$$\frac{d\mathbf{r}_\alpha}{dt} = \mathbf{v}_\alpha, \tag{3}$$

$$\frac{d\mathbf{v}_\alpha}{dt} = -\xi_\alpha \mathbf{v}_\alpha - \frac{1}{m_\alpha}\nabla_\alpha U(\mathbf{r}_1,...,\mathbf{r}_N) + \sqrt{2D'_\alpha}\mathbf{R}_\alpha(t), \tag{4}$$

where ξ_α is a friction coefficient, D'_α a diffusion coefficient and $\mathbf{R}_\alpha(t)$ a white noise satisfying $\langle \mathbf{R}_\alpha(t)\rangle = 0$ and $\langle R_{\alpha,i}(t)R_{\beta,j}(t')\rangle = \delta_{i,j}\delta_{\alpha,\beta}\delta(t-t')$ where $\alpha = 1,...,N$ labels the particles and $i = 1,...,d$ the coordinates of space. The particles interact via the potential

$$U(\mathbf{r}_1,...,\mathbf{r}_N) = \sum_{\alpha<\beta} m_\alpha m_\beta u(|\mathbf{r}_\alpha - \mathbf{r}_\beta|), \tag{5}$$

where $u(|\mathbf{r}-\mathbf{r}'|)$ is a binary potential of interaction depending only on the absolute distance between the particles. The N-body Fokker-Planck equation describing the evolution of this system is

$$\frac{\partial P_N}{\partial t} + \sum_{\alpha=1}^N \left(\mathbf{v}_\alpha \cdot \frac{\partial P_N}{\partial \mathbf{r}_\alpha} + \mathbf{F}_\alpha \cdot \frac{\partial P_N}{\partial \mathbf{v}_\alpha}\right) = \sum_{\alpha=1}^N \frac{\partial}{\partial \mathbf{v}_\alpha}\cdot\left[D'_\alpha \frac{\partial P_N}{\partial \mathbf{v}_\alpha} + \xi_\alpha P_N \mathbf{v}_\alpha\right], \tag{6}$$

where $P_N(\mathbf{r}_1,\mathbf{v}_1,...,\mathbf{r}_N,\mathbf{v}_N,t)$ is the N-body distribution function and $\mathbf{F}_\alpha = -\frac{1}{m_\alpha}\nabla_\alpha U$ is the force by unit of mass experienced by particle α. In order to obtain the canonical distribution (8) at statistical equilibrium, we must impose the Einstein relation

$$D'_\alpha = \frac{\xi_\alpha k_B T}{m_\alpha}. \tag{7}$$

This shows that the temperature T is a measure of the strength of the stochastic force in equation (4). Then, the stationary solution of the Fokker-Planck equation (6), cancelling independently [3] the advection term (l.h.s.) and the "collision" term (r.h.s.), is the canonical distribution

$$P_N(\mathbf{r}_1,\mathbf{v}_1,...,\mathbf{r}_N,\mathbf{v}_N) = \frac{1}{Z(\beta)} e^{-\beta H(\mathbf{r}_1,\mathbf{v}_1,...,\mathbf{r}_N,\mathbf{v}_N)}, \tag{8}$$

where H is the Hamiltonian

$$H = \sum_{\alpha=1}^N m_\alpha \frac{v_\alpha^2}{2} + \sum_{\alpha<\beta} m_\alpha m_\beta u(|\mathbf{r}_\alpha - \mathbf{r}_\beta|). \tag{9}$$

In the overdamped limit where $\xi_\alpha \to +\infty$, we can neglect the inertia of the particles in equation (4) and we get

$$\frac{d\mathbf{r}_\alpha}{dt} = -\mu_\alpha \nabla_\alpha U(\mathbf{r}_1,...,\mathbf{r}_N) + \sqrt{2D_\alpha}\mathbf{R}_\alpha(t), \tag{10}$$

where we have introduced the mobility $\mu_\alpha = 1/(\xi_\alpha m_\alpha)$ and the spatial diffusion coefficient $D_\alpha = D'_\alpha/\xi_\alpha^2$. The N-body Fokker-Planck equation describing the evolution of this system is

$$\frac{\partial P_N}{\partial t} = \sum_{\alpha=1}^N \frac{\partial}{\partial \mathbf{r}_\alpha}\cdot\left[D_\alpha \frac{\partial P_N}{\partial \mathbf{r}_\alpha} + \mu_\alpha P_N \frac{\partial U}{\partial \mathbf{r}_\alpha}\right], \tag{11}$$

where $P_N(\mathbf{r}_1,...,\mathbf{r}_N,t)$ is the N-body distribution in configuration space. In order to obtain the canonical distribution (14) at statistical equilibrium, we must impose the Einstein relation[1]

$$\frac{\mu_\alpha}{D_\alpha} = \frac{1}{k_B T} \equiv \beta. \tag{12}$$

Using the expression of the mobility, the Einstein relation can be rewritten

$$D_\alpha = \frac{k_B T}{\xi_\alpha m_\alpha}. \tag{13}$$

This expression can also be obtained from equation (7). Then, the stationary solution of the Fokker-Planck equation (11) is the configurational part of the canonical distribution

$$P_N(\mathbf{r}_1,...,\mathbf{r}_N) = \frac{1}{Z(\beta)} e^{-\beta U(\mathbf{r}_1,...,\mathbf{r}_N)}. \tag{14}$$

If we introduce the one and two-body probability distributions

$$P_\alpha(\mathbf{r},t) = \int P_N(\mathbf{r}_1,...,\mathbf{r}_N)\prod_{\gamma\neq\alpha} d\mathbf{r}_\gamma, \tag{15}$$

$$P_{\alpha,\beta}(\mathbf{r},\mathbf{r}',t) = \int P_N(\mathbf{r}_1,...,\mathbf{r}_N)\prod_{\gamma\neq\alpha,\beta} d\mathbf{r}_\gamma, \tag{16}$$

we find from equation (11) that the one-body distribution function (15) satisfies a kinetic equation of the form

$$\frac{\partial P_\alpha}{\partial t} = \frac{\partial}{\partial \mathbf{r}}\cdot\left[D_\alpha \frac{\partial P_\alpha}{\partial \mathbf{r}} + \mu_\alpha \sum_{\beta\neq\alpha} m_\alpha m_\beta \times \int d\mathbf{r}' P_{\alpha,\beta}(\mathbf{r},\mathbf{r}',t)\frac{\partial u}{\partial \mathbf{r}}(|\mathbf{r}-\mathbf{r}'|)\right]. \tag{17}$$

An alternative derivation of this equation is given in Appendix A. This equation is exact and takes into account statistical correlations encapsulated in the two-body distribution (16). As a result, this equation is not closed since it involves a distribution of higher order. The complete hierarchy of equations for the reduced distributions is given in [2,3].

[1] For a multi-components system, a necessary condition for the Fokker-Planck equation (11) to admit a stationary solution is that the ratio μ_α/D_α be independent on α. This ratio is identified with the inverse temperature β.

2.2 The potential energy tensor

We introduce the potential energy tensor[2]

$$W_{ij} = -\sum_{\alpha}\sum_{\beta\neq\alpha} m_\alpha m_\beta \times \int d\mathbf{r}d\mathbf{r}' P_{\alpha,\beta}(\mathbf{r},\mathbf{r}',t) x_i \frac{\partial u}{\partial x_j}(|\mathbf{r}-\mathbf{r}'|). \quad (18)$$

Since $u(|\mathbf{r}-\mathbf{r}'|)$ depends only on the absolute distance between the particles we get

$$W_{ij} = -\sum_{\alpha\neq\beta} m_\alpha m_\beta \int d\mathbf{r}d\mathbf{r}' P_{\alpha,\beta}(\mathbf{r},\mathbf{r}',t) \times x_i \frac{u'(|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}(x_j - x'_j). \quad (19)$$

Interchanging the dummy variables α, β and $\mathbf{r}, \mathbf{r}'$, we have equivalently

$$W_{ij} = \sum_{\alpha\neq\beta} m_\alpha m_\beta \int d\mathbf{r}d\mathbf{r}' P_{\alpha,\beta}(\mathbf{r},\mathbf{r}',t) \times x'_i \frac{u'(|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}(x_j - x'_j). \quad (20)$$

Summing equations (19) and (20), we obtain

$$W_{ij} = -\frac{1}{2}\sum_{\alpha\neq\beta} m_\alpha m_\beta \times \int d\mathbf{r}d\mathbf{r}' P_{\alpha,\beta}(\mathbf{r},\mathbf{r}',t) \frac{u'(|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}(x_i - x'_i)(x_j - x'_j). \quad (21)$$

Under this form, the potential energy tensor is manifestly symmetric

$$W_{ij} = W_{ji}. \quad (22)$$

The trace of the potential energy tensor is

$$W_{ii} = -\frac{1}{2}\sum_{\alpha\neq\beta} m_\alpha m_\beta \times \int d\mathbf{r}d\mathbf{r}' P_{\alpha,\beta}(\mathbf{r},\mathbf{r}',t) u'(|\mathbf{r}-\mathbf{r}'|)|\mathbf{r}-\mathbf{r}'|. \quad (23)$$

2.3 The exact Virial theorem

We introduce the tensor of inertia[3]

$$I_{ij}(t) = \frac{1}{\xi}\sum_{\alpha}\int P_\alpha(\mathbf{r},t)\xi_\alpha m_\alpha x_i x_j d\mathbf{r}, \quad (24)$$

where $\xi \equiv \frac{1}{N}\sum_\alpha \xi_\alpha$ is the average friction coefficient. From equation (17), we obtain

$$\xi \dot{I}_{ij} = -\sum_{\alpha}\int d\mathbf{r}(x_i\delta_{kj} + x_j\delta_{ki})\Big[k_B T\frac{\partial P_\alpha}{\partial x_k} + \sum_{\beta\neq\alpha} m_\alpha m_\beta \int d\mathbf{r}' P_{\alpha,\beta}(\mathbf{r},\mathbf{r}',t)\frac{\partial u}{\partial x_k}(|\mathbf{r}-\mathbf{r}'|)\Big], \quad (25)$$

where we have used an integration by parts (the boundary term cancels out since the current of diffusion vanishes on the box due to the conservation of the normalization condition $\int P_\alpha d\mathbf{r} = 1$). On the other hand,

$$-\int (x_i\delta_{kj} + x_j\delta_{ki})\frac{\partial P_\alpha}{\partial x_k}d\mathbf{r} = 2\delta_{ij} - \oint P_\alpha(x_i dS_j + x_j dS_i), \quad (26)$$

and

$$-\sum_{\alpha}\int d\mathbf{r}\ (x_i\delta_{kj} + x_j\delta_{ki})\sum_{\beta\neq\alpha} m_\alpha m_\beta \times \int d\mathbf{r}' P_{\alpha,\beta}(\mathbf{r},\mathbf{r}',t)\frac{\partial u}{\partial x_k}(|\mathbf{r}-\mathbf{r}'|) = 2W_{ij}. \quad (27)$$

We thus obtain

$$\xi\dot{I}_{ij} = 2Nk_BT\delta_{ij} + 2W_{ij} - \sum_\alpha k_B T\oint P_\alpha(x_i dS_j + x_j dS_i), \quad (28)$$

which is the general expression of the Virial theorem for a Brownian gas of particles in interaction in the overdamped limit. Introducing the local pressure

$$p(\mathbf{r},t) = \sum_\alpha P_\alpha(\mathbf{r},t)k_BT, \quad (29)$$

the Virial theorem can be rewritten

$$\xi\dot{I}_{ij} = 2Nk_BT\delta_{ij} + 2W_{ij} - \oint p(x_i dS_j + x_j dS_i). \quad (30)$$

The scalar Virial theorem is obtained by contracting the indices leading to

$$\frac{1}{2}\xi\dot{I} = dNk_BT + W_{ii} - \oint p\mathbf{r}\cdot d\mathbf{S}, \quad (31)$$

where

$$I(t) = \frac{1}{\xi}\sum_\alpha\int P_\alpha(\mathbf{r},t)\xi_\alpha m_\alpha r^2 d\mathbf{r}, \quad (32)$$

is the generalized moment of inertia (including friction coefficients). If the pressure is constant on the boundary of the domain, we have

$$\oint p\mathbf{r}\cdot d\mathbf{S} = p_b\oint \mathbf{r}\cdot d\mathbf{S} = p_b\int \nabla\cdot\mathbf{r}\ d\mathbf{r} = dp_bV. \quad (33)$$

More generally, we introduce the notation

$$P = \frac{1}{dV}\oint p\mathbf{r}\cdot d\mathbf{S}, \quad (34)$$

[2] This is the usual N-body potential energy tensor [8] averaged over the noise.

[3] This expression incorporates the friction coefficients ξ_α. This generalization proves to be necessary in order to obtain a closed expression of the Virial theorem for a multi-components system. For identical particles, the expression (24) coincides with the usual tensor of inertia of a N-body system [8] averaged over the noise.

which can be identified with a kinetic pressure. Then, the scalar Virial theorem can be written

$$\frac{1}{2}\xi\dot{I} = dNk_BT + W_{ii} - dPV. \quad (35)$$

At equilibrium ($\dot{I} = 0$), we have

$$dNk_BT + W_{ii} - dPV = 0. \quad (36)$$

In the absence of interaction, we recover the perfect gas law

$$PV = Nk_BT. \quad (37)$$

2.4 The gravitational potential

The gravitational potential in d dimensions is given by

$$u(\xi) = -\frac{1}{d-2}\frac{G}{\xi^{d-2}} \qquad (d \neq 2), \quad (38)$$

$$u(\xi) = G\ln\xi \qquad (d = 2), \quad (39)$$

where G is the gravity constant (whose value depends on the dimension of space). The gravitational force is given by

$$u'(\xi) = \frac{G}{\xi^{d-1}}. \quad (40)$$

Inserting this expression in equation (21), we find that the gravitational potential energy tensor reads

$$W_{ij} = -\frac{G}{2}\sum_{\alpha\neq\beta} m_\alpha m_\beta \times \int d\mathbf{r}d\mathbf{r}' P_{\alpha,\beta}(\mathbf{r},\mathbf{r}',t)\frac{(x_i - x_i')(x_j - x_j')}{|\mathbf{r}-\mathbf{r}'|^d}, \quad (41)$$

and that the trace of the potential energy tensor is

$$W_{ii} = -\frac{G}{2}\sum_{\alpha\neq\beta} m_\alpha m_\beta \int d\mathbf{r}d\mathbf{r}' P_{\alpha,\beta}(\mathbf{r},\mathbf{r}',t)\frac{1}{|\mathbf{r}-\mathbf{r}'|^{d-2}}. \quad (42)$$

For $d \neq 2$, we have

$$W_{ii} = (d-2)W, \quad (43)$$

where W is the usual potential energy [8] averaged over the noise. In that case, the scalar Virial theorem (35) can be written

$$\frac{1}{2}\xi\dot{I} = dNk_BT + (d-2)W - dPV. \quad (44)$$

For $d = 2$ a "miracle" occurs and we get the simple exact result

$$W_{ii} = -\frac{G}{2}\sum_{\alpha\neq\beta} m_\alpha m_\beta, \quad (45)$$

where we recall that $\sum_{\alpha\neq\beta} = \sum_{\alpha=1}^N \sum_{\beta\neq\alpha}$. We note that this expression, contrary to equation (43), does not depend on the configuration of the system but only on the masses of the particles. We define a critical temperature

$$k_BT_c = \frac{G\sum_{\alpha\neq\beta} m_\alpha m_\beta}{4N}. \quad (46)$$

For equal mass particles, we have

$$k_BT_c = (N-1)\frac{Gm^2}{4}. \quad (47)$$

Then, the scalar Virial theorem (35) in $d = 2$ can be written

$$\frac{1}{4}\xi\dot{I} = Nk_B(T - T_c) - PV. \quad (48)$$

At equilibrium, in a bounded domain, we get the exact equation of state

$$PV = Nk_B(T - T_c), \quad (49)$$

where P is the kinetic pressure (34). In Section 3, we shall obtain this equation of state directly from the partition function and deduce, by identification, that P also represents the thermodynamical pressure. Since $P \geq 0$, we conclude that a necessary condition for the system to be at statistical equilibrium is that $T \geq T_c$. For $T < T_c$, we cannot have statistical equilibrium since the pressure would be negative. In fact, for $T < T_c$, we have $\dot{I} \leq \epsilon < 0$ so that $I(t) = 0$ at a finite time $t = t_{end}$ implying that the system forms a Dirac peak containing all the mass in a finite time. Therefore, statistical equilibrium is not possible for $T < T_c$. However, the condition $T \geq T_c$ does not guarantee statistical equilibrium. We shall see in Section 4 that statistical equilibrium is possible only for $T \geq T_*$ where T_* is strictly larger than T_c. For $T_c < T < T_*$, the system collapses (or has a peculiar temporal behaviour) so that equation (49), corresponding to $\dot{I} = 0$, is never satisfied.

In an infinite domain, assuming that initially the particles are in a confined region of space, the pressure at infinity vanishes ($P = 0$) and the scalar Virial theorem (48) reduces to

$$\frac{1}{4}\xi\dot{I} = Nk_B(T - T_c). \quad (50)$$

This equation is readily integrated leading to

$$I(t) = \frac{4Nk_B}{\xi}(T - T_c)t + I(0). \quad (51)$$

For $T < T_c$, the moment of inertia vanishes ($I = 0$) at

$$t_{end} = \frac{I(0)\xi}{4Nk_B(T_c - T)}, \quad (52)$$

implying that the system forms a Dirac peak containing the whole mass in a finite time t_{end}. For $T > T_c$, $I(t) \to +\infty$ for $t \to +\infty$ indicating that the system evaporates.

2.5 The diffusion coefficient in d = 2

In this section, we restrict ourselves to a one component system. In that case,

$$I = \int \rho r^2 d\mathbf{r}, \tag{53}$$

is the usual moment of inertia with $\rho = NmP_\alpha$ for any $\alpha = 1, ..., N$. The mean squared displacement of a particle (any) is given by

$$\langle r^2 \rangle = \frac{\int \rho r^2 d\mathbf{r}}{M} = \frac{I}{M}. \tag{54}$$

Thus, from equation (50), we obtain

$$\frac{d\langle r^2 \rangle}{dt} = \frac{4k_B}{\xi m}(T - T_c), \tag{55}$$

where T_c is given by equation (47). After integration, we get

$$\langle r^2 \rangle = \frac{4k_B}{\xi m}(T - T_c)t + \langle r^2 \rangle_0. \tag{56}$$

The diffusion coefficient of the particle is defined by

$$\langle r^2 \rangle \sim 4D(T)t, \qquad (t \to +\infty). \tag{57}$$

Therefore, it is given by the exact expression

$$D(T) = \frac{k_B}{\xi m}(T - T_c). \tag{58}$$

In the absence of gravity ($G = T_c = 0$), or for high temperatures ($T \gg T_c$) where gravitational attraction becomes negligible with respect to thermal motion, we recover the usual expression of the diffusion coefficient given by the Einstein relation

$$D = \frac{k_B T}{\xi m}. \tag{59}$$

However, for smaller temperatures, gravitational effects come into play and the expression of the diffusion coefficient is modified. Interestingly, there exists a critical temperature T_c at which the diffusion coefficient vanishes[4]. For $T > T_c$ the diffusion coefficient is positive so that the system evaporates. For $T < T_c$ the diffusion coefficient becomes negative implying finite time collapse to a Dirac peak containing the whole mass in a time

$$t_{end} = \frac{m\xi\langle r^2 \rangle_0}{4k_B(T_c - T)}. \tag{60}$$

This time behaves like $(T_c - T)^{-1}$ for *any* $T < T_c$ and diverges at the critical point T_c. At $T = T_c$ the Dirac peak is formed in infinite time. These different regimes have been studied in [12,16,19] by solving the (mean field) Smoluchowski-Poisson system in two dimensions.

[4] This is physically different from the vanishing of the diffusion coefficient in the case of colloids [29] which is due to the close packing of the particles (steric hindrance) in the glassy phase (the particles cannot move freely) while in the present situation we have a collapse due to an attractive potential.

2.6 The mean field approximation

The preceding results are exact for self-gravitating Brownian particles in $d = 2$ dimensions, whatever the number N of constituents. In particular, they take into account statistical correlations. However, in most works on self-gravitating systems, one usually considers a mean field approximation where the equations of the problem can be simplified. In general, this approximation is valid for $N \gg 1$. In this section, we briefly describe how the mean field approximation can be implemented and how the results are modified.

The essence of the mean field approximation is to assume that the two-body probability distribution can be written as the product of two one-body probability distributions according to

$$P_{\alpha,\beta}(\mathbf{r}, \mathbf{r}', t) = P_\alpha(\mathbf{r}, t)P_\beta(\mathbf{r}', t). \tag{61}$$

This approximation allows one to close the hierarchy of equations at the level of equation (17). For systems with long-range interactions, it can be shown that this approximation is exact in a proper thermodynamic limit $N \to +\infty$ with $\eta = \beta N u_*$ and $\epsilon = E/(u_* N^2)$ fixed, where u_* is the typical value of the binary potential. For the gravitational potential in d dimensions where $u_* = Gm^2/R^{d-2}$, the thermodynamic limit corresponds to $N \to +\infty$ in such a way that the dimensionless temperature $\eta = \beta GMm/R^{d-2}$ and the dimensionless energy $\epsilon = ER^{d-2}/GM^2$ are fixed. One can always rescale the quantities of the problem so that the coupling constant (G in gravity) scales like $u_* \sim 1/N$ while $E \sim N$, $T \sim 1$ and $V \sim 1$ [2]. In this limit, the factorization (61) is valid up to terms of order $1/N$. Since we consider a large number limit, we can also extend the sum in equation (17) over all the particles. This yields

$$\frac{\partial P_\alpha}{\partial t} = \frac{\partial}{\partial \mathbf{r}} \cdot \Big[D_\alpha \frac{\partial P_\alpha}{\partial \mathbf{r}} + \mu_\alpha m_\alpha P_\alpha(\mathbf{r}, t) \times \nabla \sum_\beta m_\beta \int d\mathbf{r}' P_\beta(\mathbf{r}', t) u(|\mathbf{r} - \mathbf{r}'|) \Big]. \tag{62}$$

If we introduce the density

$$\rho(\mathbf{r}, t) = \sum_\alpha P_\alpha(\mathbf{r}, t) m_\alpha, \tag{63}$$

we get

$$\frac{\partial P_\alpha}{\partial t} = \frac{\partial}{\partial \mathbf{r}} \cdot \Big[D_\alpha \frac{\partial P_\alpha}{\partial \mathbf{r}} + \mu_\alpha m_\alpha P_\alpha(\mathbf{r}, t) \times \nabla \int d\mathbf{r}' \rho(\mathbf{r}', t) u(|\mathbf{r} - \mathbf{r}'|) \Big]. \tag{64}$$

This equation can be rewritten in the form of a mean field Smoluchowski equation

$$\frac{\partial P_\alpha}{\partial t} = \frac{\partial}{\partial \mathbf{r}} \cdot \Big[D_\alpha \frac{\partial P_\alpha}{\partial \mathbf{r}} + \mu_\alpha m_\alpha P_\alpha(\mathbf{r}, t) \nabla \Phi \Big], \tag{65}$$

where the potential is determined by the density according to

$$\Phi(\mathbf{r},t)=\int \rho(\mathbf{r}',t)u(|\mathbf{r}-\mathbf{r}'|)d\mathbf{r}'. \quad (66)$$

For the gravitational potential, the preceding equation is equivalent to the Poisson equation

$$\Delta\Phi=S_dG\rho. \quad (67)$$

Therefore, in the mean field approximation, we have to solve the multi-components Smoluchowski-Poisson system (65)–(67). The steady state of equation (65) is the mean field Boltzmann distribution

$$P_\alpha(\mathbf{r})=A_\alpha e^{-\beta m_\alpha\Phi(\mathbf{r})}. \quad (68)$$

The single component Smoluchowski-Poisson system has been studied in [11–17,19–21] and the two components Smoluchowski-Poisson system has been studied in [18].

To establish the expression of the Virial theorem in the mean field approximation, we can use a procedure similar to the one developed in Section 2.3. The only change is the factorization (61) and the replacement of $\sum_{\beta\neq\alpha}$ by $\sum_\beta$. Thus, the previous relations remain valid provided that W_{ij} is replaced by

$$W_{ij}^{MF}=-\int \rho(\mathbf{r},t)x_i\frac{\partial\Phi}{\partial x_j}, \quad (69)$$

where $\Phi(\mathbf{r},t)$ is given by equation (66). In particular, for the gravitational interaction we have

$$W_{ij}^{MF}=-\frac{G}{2}\int d\mathbf{r}d\mathbf{r}'\rho(\mathbf{r},t)\rho(\mathbf{r}',t)\frac{(x_i-x_i')(x_j-x_j')}{|\mathbf{r}-\mathbf{r}'|^d}, \quad (70)$$

and the trace of the potential energy tensor is

$$W_{ii}^{MF}=-\frac{G}{2}\int d\mathbf{r}d\mathbf{r}'\frac{\rho(\mathbf{r},t)\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|^{d-2}}. \quad (71)$$

For $d\neq 2$,

$$W_{ii}^{MF}=(d-2)W^{MF}, \quad (72)$$

where W^{MF} is the mean field potential energy

$$W^{MF}=\frac{1}{2}\int \rho\Phi d\mathbf{r}, \quad (73)$$

with

$$\Phi(\mathbf{r},t)=-\frac{G}{d-2}\int\frac{\rho(\mathbf{r},t)}{|\mathbf{r}-\mathbf{r}'|^{d-2}}d\mathbf{r}'. \quad (74)$$

For $d=2$, we get

$$W_{ii}^{MF}=-\frac{G}{2}\int d\mathbf{r}d\mathbf{r}'\rho(\mathbf{r},t)\rho(\mathbf{r}',t)=-\frac{GM^2}{2}. \quad (75)$$

Therefore, in the mean field approximation, the scalar Virial theorem of a self-gravitating Brownian gas in $d=2$ can be written as in equation (48) with the critical temperature

$$k_BT_c^{MF}=\frac{GM^2}{4N}. \quad (76)$$

For equal mass particles it reduces to

$$k_BT_c^{MF}=\frac{GNm^2}{4}. \quad (77)$$

These expressions can be directly obtained from equations (46) and (47) by replacing $\sum_{\alpha\neq\beta}$ by $\sum_\alpha\sum_\beta$ in equation (46) or by replacing $N-1$ by N in equation (47) since the mean field approximation is valid for $N\gg 1$. We note that the mean field results are relatively close to the exact results even for a moderate number of particles. Equation (48) is always valid and finite N effects just slightly shift the critical temperature T_c. Since $N-1=N(1-1/N)$, the correction is of order $1/N$, which is precisely the domain of validity of the factorization hypothesis (61) as shown in [2,3] at a more general level. This corroborates the observation that the mean field approximation provides a good description of systems with long-range interactions such as self-gravitating systems.

3 The exact equation of state

In this section, we derive the exact equation of state of a self-gravitating gas in two dimensions. We extend the derivation given by Salzberg [32] and Padmanabhan [4] in two respects: (1) we consider a multi-components system while the previous authors assume that the particles have the same mass; (2) we treat both the canonical and the microcanonical ensembles while the previous authors only consider the canonical ensemble. For comparison, we also discuss the case of a two-dimensional plasma made of electric charges [31].

3.1 Canonical approach

For the gravitational interaction in two dimensions

$$U(\mathbf{r}_1,...,\mathbf{r}_N)=G\sum_{i<j}m_im_j\ln|\mathbf{r}_i-\mathbf{r}_j|, \quad (78)$$

the configurational part of the partition function in the canonical ensemble is given by

$$Z(\beta,V)=\int e^{-\beta G\sum_{i<j}m_im_j\ln|\mathbf{r}_i-\mathbf{r}_j|}\prod_{k=1}^N d\mathbf{r}_k. \quad (79)$$

To avoid the divergence of the partition function at large distances, we assume that the system is enclosed within a box of radius R. The following calculations also assume implicitly that the partition function converges at small distances. The existence of statistical equilibrium states will be discussed in Section 4. Using a trick due to

Salzberg [32], we set $\mathbf{y} = \mathbf{r}/R$. Then, the partition function can be rewritten

$$Z(\beta, V) = R^{2N} e^{-\beta G \ln R \sum_{i<j} m_i m_j} \times \int e^{-\beta G \sum_{i<j} m_i m_j \ln |\mathbf{y}_i - \mathbf{y}_j|} \prod_{k=1}^{N} d\mathbf{y}_k, \quad (80)$$

where the last integral is now *independent* on R. From this expression, we find that

$$\frac{\partial Z}{\partial R} = \frac{2N}{R} \left[1 - \frac{\beta G}{2N} \sum_{i<j} m_i m_j \right] Z(\beta, R). \quad (81)$$

The thermodynamic pressure is defined by

$$P = \frac{1}{\beta} \frac{\partial \ln Z}{\partial V}, \qquad V = \pi R^2. \quad (82)$$

From equations (82) and (81), we obtain the exact equation of state of a two-dimensional multi-components self-gravitating gas

$$P = \frac{N}{\beta V} \left[1 - \frac{\beta G}{2N} \sum_{i<j} m_i m_j \right]. \quad (83)$$

This can be written

$$PV = N k_B (T - T_c), \quad (84)$$

with the critical temperature

$$k_B T_c = \frac{G \sum_{i \neq j} m_i m_j}{4N}. \quad (85)$$

For equal mass particles, we have

$$k_B T_c = (N-1) \frac{G m^2}{4}. \quad (86)$$

This returns the equation of state (49) obtained by the kinetic approach. Note that for a single species system in the mean field approximation, the equation of state (84) can also be obtained by solving the Boltzmann-Poisson equation and computing the pressure at the edge of the box [33].

3.2 Microcanonical approach

The Hamiltonian of a self-gravitating system in two dimensions is

$$H = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 + G \sum_{i<j} m_i m_j \ln |\mathbf{r}_i - \mathbf{r}_j|. \quad (87)$$

The density of states in the microcanonical ensemble is defined by

$$g(E, R) = \int \delta(E - H) \prod_{k=1}^{N} d\mathbf{r}_k d\mathbf{v}_k. \quad (88)$$

With the transformation $\mathbf{y} = \mathbf{r}/R$, it can be written

$$g(E, R) = R^{2N} g(E', 1), \quad (89)$$

with

$$E' = E - G \ln R \sum_{i<j} m_i m_j. \quad (90)$$

The entropy is defined by

$$S(E, R) = k_B \ln g(E, R). \quad (91)$$

According to equation (89), we have

$$S(E, R) = 2N k_B \ln R + S(E', 1). \quad (92)$$

The temperature and the pressure are given by

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E} \right)_{N,V}, \qquad P = T \left(\frac{\partial S}{\partial V} \right)_{N,E}, \quad (93)$$

with $V = \pi R^2$. From equations (92), (90) and (93)-a we find that

$$\left(\frac{\partial S}{\partial R} \right)_{N,E} = \frac{2N k_B}{R} - \frac{G}{TR} \sum_{i<j} m_i m_j. \quad (94)$$

Using equation (93)-b, we obtain

$$PV = N k_B \left[T(E) - \frac{G}{2N k_B} \sum_{i<j} m_i m_j \right]. \quad (95)$$

This returns the expression (84) obtained in the canonical ensemble with the critical temperature (85) and (86). Therefore, the equations of state coincide in the two ensembles for any number of particles. This result was not obvious at first sights. For two-dimensional self-gravitating systems, the microcanonical and canonical ensembles are equivalent at the thermodynamic limit with $N \to +\infty$[5] but they are not equivalent for finite values of N. For example, we shall explicitly show in Section 5 that the caloric curves for $N = 2$ calculated in the microcanonical and canonical ensembles differ. Yet, the equation of state is the same for any N.

3.3 The case of electric charges

The case of electric charges is obtained by taking $G = -1$ and by making the substitution $m \leftrightarrow q$. Therefore, the energy of interaction in two dimensions reads

$$U(\mathbf{r}_1, ..., \mathbf{r}_N) = - \sum_{i<j} q_i q_j \ln |\mathbf{r}_i - \mathbf{r}_j|. \quad (96)$$

[5] This is because the series of equilibria of a two-dimensional self-gravitating gas does not present turning points contrary to 3D self-gravitating systems; see, e.g., [12].

The previous results remain valid for electric charges instead of point masses, provided that we make the above mentioned substitutions. In particular, equations (83) and (95) are now replaced by

$$P=\frac{Nk_BT}{V}\left[1+\frac{\beta}{2N}\sum_{i<j}q_iq_j\right]. \quad (97)$$

Using the condition of electroneutrality

$$\sum_{i=1}^N q_i=0, \quad (98)$$

we find that

$$\sum_{i<j}q_iq_j=\frac{1}{2}\sum_{i\neq j}q_iq_j$$
$$=\frac{1}{2}\sum_{i=1}^N q_i\left(\sum_{j=1}^N q_j-q_i\right)=-\frac{1}{2}\sum_{i=1}^N q_i^2. \quad (99)$$

Therefore, the equation of state of a neutral two-dimensional plasma can be written

$$PV=Nk_B(T-T_c), \quad (100)$$

with the critical temperature

$$k_BT_c=\frac{\sum_{i=1}^N q_i^2}{4N}. \quad (101)$$

If the plasma consists in $N/2$ charges $+e$ and $N/2$ charges $-e$, we get

$$k_BT_c=\frac{e^2}{4}. \quad (102)$$

These results have been first derived by Salzberg and Prager [31] in the canonical ensemble. The calculations of Section 3.2 show that they can also be derived in the microcanonical ensemble. Since $P\geq 0$, a necessary condition for the system to be at statistical equilibrium is that $T\geq T_c$. However, this condition does not guarantee statistical equilibrium. We shall see in the sequel that statistical equilibrium exists only for $T\geq T_*=2T_c$. For $T<T_*$ there is no equilibrium state and the system collapses. This leads to the formation of $N/2$ pairs $(+,-)$ corresponding to non interacting "atoms". By contrast, for $T>T_*$ the system is fully ionized. For comparison, a gas of self-gravitating Brownian particles collapses to a single Dirac peak containing the N particles for $T<T_*=\frac{N}{N-1}T_c$ while it remains diffuse for $T>T_*$. Since the collapse in plasma physics leads to the formation of individual pairs, we understand qualitatively why the collapse temperature $k_BT_*=e^2/2$ does not depend on N (it corresponds to the condition to form *one* pair). By contrast, since the collapse in gravity leads to the formation of a single Dirac peak containing all the mass, the collapse temperature $T_*=NGm^2/4$ depends on N (it corresponds to the condition to form a cluster of N particles). Finally, it is easy to determine the critical temperature of a non neutral 2D plasma consisting in N_+ charges $+e$ and N_- charges $-e$. From, equations (97) and (100), we find

$$k_BT_c=\frac{e^2}{4}\left[1-\frac{(N_+-N_-)^2}{N}\right]. \quad (103)$$

4 Existence of statistical equilibrium in the canonical ensemble

For a finite two-dimensional self-gravitating system, there exists statistical equilibrium states for any value of the energy in the microcanonical ensemble (Hamiltonian systems). Indeed, if the system is enclosed within a box, the density of states $g(E)$ is convergent for any E. By contrast, in the canonical ensemble (Brownian systems) equilibrium states exist only for sufficiently high temperatures $T>T_*$. Although the existence of a collapse temperature T_* is well-known, there is some ambiguity in the literature concerning its precise value. In particular, we will show that the collapse temperature T_* does not exactly coincide with the critical temperature T_c introduced previously. We will also show that the collapse temperature T_* is difficult to determine for a multi-components system while it takes a simple expression when the particles have the same mass m.

4.1 Statistical equilibrium state for $T>T_1$

In this section, using the arithmetic-geometric mean inequality, we show that the partition function $Z(\beta)$ is convergent for $T>T_1$. We extend the method developed by Kiessling [34] to the case of a multi-components system. First, we note that

$$e^{-\beta G\sum_{i<j}m_im_j\ln|\mathbf{r}_i-\mathbf{r}_j|}=e^{-\frac{1}{2}\beta G\sum_i\sum_{j\neq i}m_im_j\ln|\mathbf{r}_i-\mathbf{r}_j|}$$
$$=\prod_i e^{-\frac{1}{2}\beta G\sum_{j\neq i}m_im_j\ln|\mathbf{r}_i-\mathbf{r}_j|}$$
$$=\prod_i\left[e^{-\frac{N}{2}\beta G\sum_{j\neq i}m_im_j\ln|\mathbf{r}_i-\mathbf{r}_j|}\right]^{1/N}. \quad (104)$$

Now, by the arithmetic-geometric mean inequality

$$\frac{1}{N}\sum_i a_i\geq\prod_i a_i^{1/N}, \quad (105)$$

we get

$$e^{-\beta G\sum_{i<j}m_im_j\ln|\mathbf{r}_i-\mathbf{r}_j|}\leq\frac{1}{N}\sum_i e^{-\frac{N}{2}\beta G\sum_{j\neq i}m_im_j\ln|\mathbf{r}_i-\mathbf{r}_j|}$$
$$=\frac{1}{N}\sum_i\prod_{j\neq i}e^{-\frac{N}{2}\beta Gm_im_j\ln|\mathbf{r}_i-\mathbf{r}_j|}. \quad (106)$$

From this inequality, we find that the configurational part of the partition function satisfies

$$Z \leq \frac{1}{N}\sum_i \int d\mathbf{r}_i \prod_{j\neq i}\int d\mathbf{r}_j e^{-\frac{N}{2}\beta G m_i m_j \ln|\mathbf{r}_i-\mathbf{r}_j|}. \tag{107}$$

The integrand is maximum for $\mathbf{r}_i = \mathbf{0}$ yielding

$$\begin{aligned} Z &\leq \frac{1}{N}\sum_i \int d\mathbf{r}_i \prod_{j\neq i}\int e^{-\frac{N}{2}\beta G m_i m_j \ln x}\, d\mathbf{x} \\ &= \frac{\pi R^2}{N}\sum_i \prod_{j\neq i}\int x^{-\frac{N}{2}\beta G m_i m_j}\, d\mathbf{x} \\ &= \frac{2\pi^2 R^2}{N}\sum_i \prod_{j\neq i}\int_0^R x^{1-\frac{N}{2}\beta G m_i m_j} dx. \end{aligned} \tag{108}$$

If

$$k_B T > \frac{N G m_i m_j}{4} \tag{109}$$

for all i, j with $i \neq j$, then all the integrals converge and the partition function is finite. Therefore, a sufficient condition for the existence of statistical equilibrium is that $T > T_1$ with

$$k_B T_1 = \frac{N G m_I m_{II}}{4}, \tag{110}$$

where I and II denote the two most massive particles in the system. In that case, we obtain the following bound on the partition function

$$Z \leq \frac{\pi R^2}{N}\sum_i \prod_{j\neq i}\frac{\pi R^{2-\frac{N}{2}\beta G m_i m_j}}{1-\frac{N}{4}\beta G m_i m_j}. \tag{111}$$

For equal mass particles, a sufficient condition for the existence of statistical equilibrium is that $T > T_1$ with

$$k_B T_1 = \frac{N G m^2}{4}. \tag{112}$$

In that case, we recover the bound on the partition function given by Kiessling [34]:

$$Z \leq \pi R^2 \left(\frac{\pi R^{2-\frac{N}{2}\beta G m^2}}{1-\frac{N}{4}\beta G m^2}\right)^{N-1}. \tag{113}$$

4.2 Collapse for $\mathbf{T < T_2}$

We now show that the partition function is divergent for $T < T_2$. We first rewrite the configurational partition function in the form

$$Z = \int d\mathbf{r}_1 ... d\mathbf{r}_N \prod_{i\neq j}\frac{1}{|\mathbf{r}_i-\mathbf{r}_j|^{\frac{1}{2}\beta G m_i m_j}}. \tag{114}$$

We select a particle (allowed to move over the whole box) and approach the $N-1$ other particles at a distance ϵ of the first [34,35]. The contribution of this configuration to the partition function behaves like

$$\begin{aligned} Z &\sim R^2 \epsilon^{2(N-1)} \prod_{i\neq j}\left(\frac{1}{\epsilon}\right)^{\frac{1}{2}\beta G m_i m_j} \\ &= R^2 \left(\frac{1}{\epsilon}\right)^{\sum_{i\neq j}\frac{1}{2}\beta G m_i m_j - 2(N-1)}. \end{aligned} \tag{115}$$

Considering now the limit $\epsilon \to 0$, we see that the partition function diverges if $T < T_2$ with

$$k_B T_2 = \frac{G\sum_{i\neq j} m_i m_j}{4(N-1)}. \tag{116}$$

For equal mass particles, we find that

$$k_B T_2 = N\frac{G m^2}{4}. \tag{117}$$

4.3 The collapse temperature $\mathbf{T_*}$

In conclusion, the partition function converges for $T > T_1$ and diverges for $T < T_2$. For $T_2 < T < T_1$, we have no result in the general case. However, for equal mass particles, we have $T_1 = T_2 = T_*$ with

$$k_B T_* = N\frac{G m^2}{4}. \tag{118}$$

Therefore, the partition function converges for $T > T_*$ and diverges for $T < T_*$. For $T > T_*$, we have a diffuse gas and for $T < T_*$ the most probable distribution is a Dirac peak containing all the particles. We note that T_* *differs* by a factor $1 - 1/N$ from the critical temperature T_c appearing in the equation of state (84). Indeed,

$$T_* = \frac{N}{N-1}T_c. \tag{119}$$

This point has been overlooked in the literature. In Section 5, we shall provide a clear evidence of this difference by considering explicitly the case $N = 2$ where the partition function can be calculated exactly. Finally, we note that the pressure (84) at the collapse temperature T_* is equal to

$$PV = k_B T_*, \qquad (T = T_*). \tag{120}$$

This corresponds to the ideal pressure due to $N_* = 1$ effective particle (the Dirac peak) containing the whole mass when the system has collapsed.

For a multi-components system, the collapse temperature T_* seems difficult to obtain. The case of a two-components system has been considered by Sopik et al. [18] in the mean field approximation and an explicit expression of the collapse temperature has been obtained in that case.

4.4 The case of 2D plasmas

For self-gravitating systems, the partition function diverges for $T \leq T_*$ due to the collapse of *all* the particles at the same point. Thus, below T_*, the most probable structure is a Dirac peak containing the whole mass. By using an argument similar to that of equation (115), we see that if we concentrate only a fraction of the particles (e.g. a pair), we obtain a smaller collapse temperature, so that this configuration is less favorable.

By contrast, for neutral plasmas with $N/2$ charges $+e$ and $N/2$ charges $-e$, the divergence of the partition function corresponds to the formation (collapse) of individual pairs $(+,-)$. Therefore, we can obtain the collapse temperature by considering the statistical mechanics of only *one* pair ($N = 2$). This is done it the next section and the analysis yields

$$k_B T_* = \frac{e^2}{2}. \tag{121}$$

Therefore, the partition function of the N-body system converges for $T > T_*$ and diverges for $T < T_*$. For $T > T_*$, we have a fully ionized gas and for $T < T_*$ the most probable distribution is $N/2$ pairs $(+,-)$. We note that T_* *differs* by a factor 2 from the critical temperature T_c appearing in the equation of state (100). Indeed,

$$T_* = 2T_c. \tag{122}$$

Since the equilibrium states exists only for $T > T_*$, the equation of state (100) is valid only above T_*. For $T < T_*$ (including the region $T_c < T < T_*$), the partition function diverges due to the collapse of pairs of point particles of opposite sign. Finally, we note that the pressure (100) at the collapse temperature T_* is equal to

$$PV = \frac{1}{2} N k_B T_*, \qquad (T = T_*). \tag{123}$$

This is the pressure created by an *ideal* gas of $N_* = N/2$ non-interacting pairs $(+,-)$. Finally, for $T < T_c$, the Virial theorem shows that the N_* pairs collapse to a *single* point in a finite time (if the motion of the charges in the plasma is described by equation (10)) while we have no result for $T_c < T < T_*$. In that range of temperatures we probably have a homogeneous distribution of pairs.

5 The case N = 2

The statistical mechanics of two particles in gravitational interaction (binary star) was considered by Padmanabhan [4] in $d = 3$ dimensions. He showed that this "toy model" exhibits phenomena that are representative of more realistic stellar systems with a large number of particles. Here, we shall extend his analysis in $d = 2$ dimensions (the one-dimensional case is treated in Appendix C). For $N = 2$ particles, it is possible to compute the density of states and the partition function exactly. This will allow us to illustrate the preceding results on an explicit example.

5.1 The microcanonical ensemble

The Hamiltonian of a system of two particles with mass m_1 and m_2 in gravitational interaction can be written

$$H = \frac{1}{2} M V^2 + \frac{1}{2} \mu v^2 + G m_1 m_2 \ln r, \tag{124}$$

where $(\mathbf{R}, \mathbf{V})$ denote the position and the velocity of the centre of mass and $(\mathbf{r}, \mathbf{v})$ the position and the velocity of the reduced particle. On the other hand

$$M = m_1 + m_2, \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}, \tag{125}$$

denote, respectively, the total mass of the particles and the mass of the reduced particle. We shall first compute the hypersurface of phase space with energy less than E, i.e.

$$\Gamma(E) = \int_{H \leq E} d\mathbf{R} d\mathbf{V} d\mathbf{r} d\mathbf{v}. \tag{126}$$

Noting $x_i = (M/2)^{1/2} V_i$ for $i = 1, 2$ and $x_i = (\mu/2)^{1/2} v_i$ for $i = 3, 4$, the preceding quantity can be rewritten

$$\Gamma(E) = \frac{4\pi R^2}{\mu M} \int d\mathbf{r} \int_{\|\mathbf{x}\| \leq \sqrt{E - G m_1 m_2 \ln r}} d^4 x. \tag{127}$$

The last integral represents the volume of a four-dimensional hypersphere with radius $\sqrt{E - G m_1 m_2 \ln r}$. Therefore, we obtain

$$\Gamma(E) = \frac{8\pi^2 R^2}{\mu M} V_4 \int_0^{r_m} \left(E - G m_1 m_2 \ln r\right)^2 r dr, \tag{128}$$

where r_m denotes the maximum distance accessible to the reduced particle and V_4 is the volume of a four-dimensional hypersphere with unit radius. From the general expression

$$V_d = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)}, \tag{129}$$

we find that $V_4 = \pi^2/2$. The density of states in the microcanonical ensemble is given by $g(E) = d\Gamma/dE$. Using equation (128), we obtain

$$g(E) = \frac{8\pi^4 R^2}{\mu M} \int_0^{r_m} \left(E - G m_1 m_2 \ln r\right) r dr. \tag{130}$$

The range of integration r_m is such that $E - G m_1 m_2 \ln r \geq 0$ and $r \leq R$. Therefore, $r_m = e^{E/G m_1 m_2}$ if $E \leq G m_1 m_2 \ln R$ and $r_m = R$ if $E \geq G m_1 m_2 \ln R$. Introducing

$$A = 8\pi^4 G, \quad \epsilon = \frac{E}{G m_1 m_2}, \quad t = \frac{k_B T}{G m_1 m_2}, \tag{131}$$

we obtain

$$g(\epsilon) = A R^2 \int_0^{r_m} \left(\epsilon - \ln r\right) r dr, \tag{132}$$

with $r_m = e^\epsilon$ if $\epsilon \leq \ln R$ and $r_m = R$ if $\epsilon \geq \ln R$. The entropy is given by

$$S(\epsilon) = k_B \ln g(\epsilon), \tag{133}$$

and the temperature by

$$\frac{1}{T} = \frac{\partial S}{\partial E} \quad \rightarrow \frac{1}{t} = \frac{\partial (S/k_B)}{\partial \epsilon}. \tag{134}$$

Finally, the pressure is given by

$$P = T\frac{\partial S}{\partial V}, \qquad V = \pi R^2. \tag{135}$$

The caloric curve $t = t(\epsilon)$ and the equation of state in the microcanonical ensemble can be obtained from the exact expression (132) of the density of states. For $\epsilon \leq \ln R$, we obtain

$$g(\epsilon) = \frac{AR^2}{4} e^{2\epsilon}, \tag{136}$$

$$k_B T = \frac{G m_1 m_2}{2} \quad \rightarrow t = \frac{1}{2}, \tag{137}$$

$$PV = k_B T. \tag{138}$$

For $\epsilon \geq \ln R$, we obtain

$$g(\epsilon) = \frac{AR^4}{2}\left(\epsilon - \ln R + \frac{1}{2}\right), \tag{139}$$

$$t = \epsilon - \ln R + \frac{1}{2}, \tag{140}$$

$$PV = 2k_B(T - T_c), \tag{141}$$

with

$$k_B T_c = \frac{G m_1 m_2}{4}. \tag{142}$$

The caloric curve $t(\epsilon)$ in the microcanonical ensemble is represented in Figure 1.

5.2 The canonical ensemble

For $N = 2$ particles in gravitational interaction, the partition function in the canonical ensemble is given by

$$Z = \int e^{-\beta M \frac{V^2}{2}} e^{-\beta \mu \frac{v^2}{2}} e^{-\beta m_1 m_2 G \ln r} d\mathbf{R} d\mathbf{V} d\mathbf{r} d\mathbf{v}. \tag{143}$$

Integrating on $\mathbf{R}$ and on the velocities, we get

$$Z = \frac{8\pi^3 V}{\beta^2 m_1 m_2} \int_0^R r^{1-\beta G m_1 m_2}\, dr. \tag{144}$$

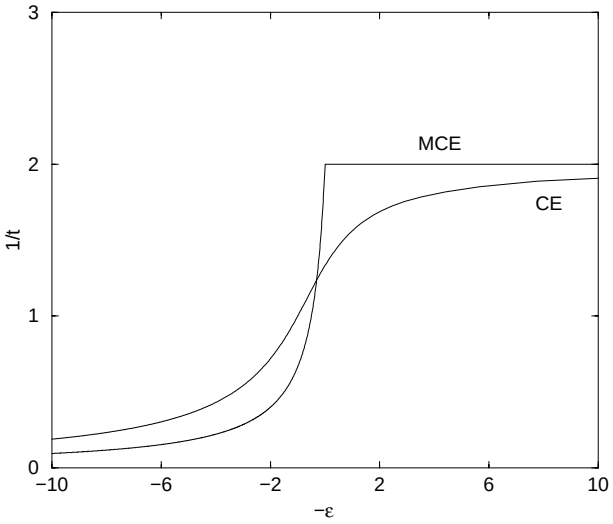

Fig. 1. Caloric curve in microcanonical (MCE) and canonical (CE) ensembles for the binary star model $N = 2$ in two-dimensions. The caloric curves are similar in the two ensembles (without negative specific heat region) so that there is no phase transition contrary to the situation in $d = 3$ [4,5]. There exists equilibrium states for all energies in MCE and for temperatures $t > t_* = 1/2$ in CE. It can be of interest to compare these curves with the caloric curve obtained within the mean field approximation valid for $N \gg 1$ which presents a similar behaviour [33,12].

The partition function is finite if, and only if, $T > T_*$ with

$$k_B T_* = \frac{G m_1 m_2}{2}. \tag{145}$$

For equal mass particles, we have

$$k_B T_* = \frac{G m^2}{2}. \tag{146}$$

These expressions agree with the general results of Section 4. For $N = 2$, we have $T_1 = T_2 = T_*$ even if the particles have different masses $m_1 \neq m_2$. For $T > T_*$, the partition function is explicitly given by

$$Z = \frac{8\pi^4}{\beta^2 m_1 m_2} \frac{R^{4-\beta G m_1 m_2}}{2 - \beta G m_1 m_2}. \tag{147}$$

This expression corresponds to the bound obtained in equation (111). The pressure can be computed from equations (82) and (147) leading to

$$PV = 2k_B(T - T_c), \tag{148}$$

with

$$k_B T_c = \frac{G m_1 m_2}{4}. \tag{149}$$

For equal mass particles, we get

$$k_B T_c = \frac{G m^2}{4}. \tag{150}$$

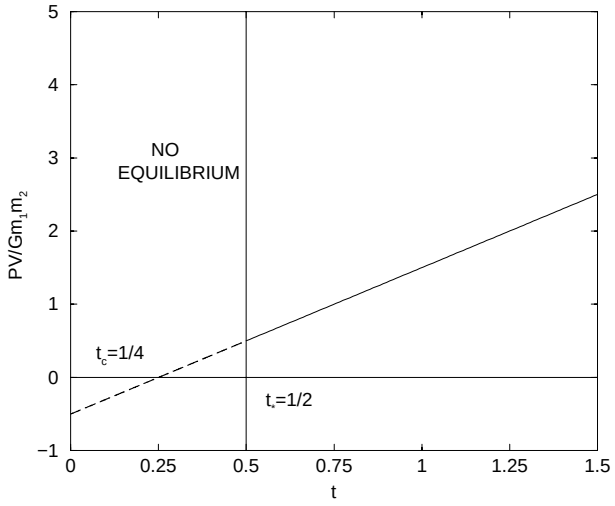

Fig. 2. Equation of state for the binary star model $N = 2$ in two-dimensions. Equilibrium states exist only for $t > t_* = 1/2$ so that the portion of the curve $t_c = 1/4 < t < t_* = 1/2$ is not physical. Note that the pressure does not vanish at $t = t_* = 1/2$. It corresponds to the pressure due to the Dirac peak (one effective particle) that forms at this critical temperature (see Sect. 4.3).

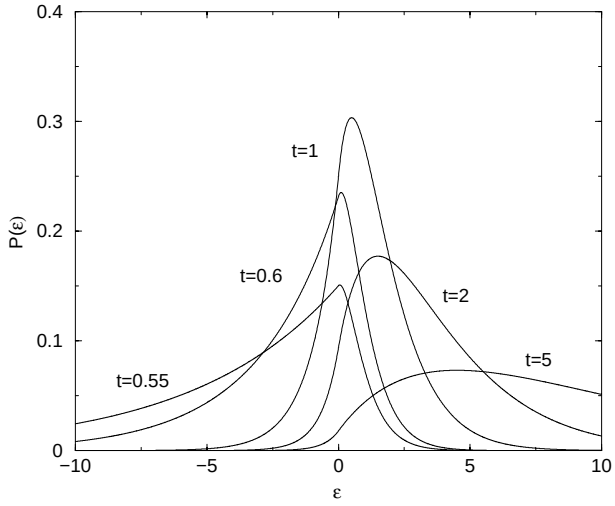

Fig. 3. Distribution of energies for the binary star model $N = 2$ in two-dimensions in the canonical ensemble. We have plotted this distribution for different values of the temperature.

These expressions are consistent with equations (84, 86). We also check explicitly that $T_* \neq T_c$ (for $N = 2$, we have $T_* = 2T_c$). Therefore, T_c is smaller than the collapse temperature T_* below which the partition function diverges. The curve $P(T)$ is plotted in Figure 2.

The caloric curve $\langle\epsilon\rangle(t)$ in the canonical ensemble can be obtained from the exact expression (147) of the partition function. The average energy is given by

$$\langle E\rangle = -\frac{\partial \ln Z}{\partial \beta} \quad \rightarrow \langle\epsilon\rangle = t^2\frac{\partial \ln Z}{\partial t}. \tag{151}$$

Using equation (147), we obtain

$$\langle\epsilon\rangle = \frac{(4t-3)t}{2t-1} + \ln R. \tag{152}$$

The caloric curve $\langle\epsilon\rangle(t)$ in the canonical ensemble is represented in Figure 1 and is compared with the caloric curve in the microcanonical ensemble. We note that, for $N = 2$, the specific heats $C = d\langle E\rangle/dT$ diverges like $\propto (T-T_*)^{-2}$ at $T = T_*$. By contrast, in the limit $N \to +\infty$ where a mean field approximation can be implemented, we find from equation (40) of Sire and Chavanis [12] that $C \propto (T - T_*)^{-1}$.

Finally, the distribution of energies at temperature T in the canonical ensemble is given by

$$P(E) = \frac{1}{Z(T)} g(E) e^{-\beta E}. \tag{153}$$

Using $P(E)dE = P(\epsilon)d\epsilon$, this can be rewritten

$$\frac{P(\epsilon)}{Gm_1m_2} = \frac{1}{Z(t)} g(\epsilon) e^{-\epsilon/t}. \tag{154}$$

Using equations (136, 139) and (147), we find that

$$P(\epsilon) = \frac{2t-1}{4t^3} e^{(2-1/t)\epsilon}, \qquad (\epsilon \leq 0), \tag{155}$$

$$P(\epsilon) = \frac{2t-1}{4t^3}(2\epsilon+1) e^{-\epsilon/t}, \qquad (\epsilon \geq 0). \tag{156}$$

We have taken $R = 1$ to simplify the expressions. The distribution of energies is represented in Figure 3 for different temperatures.

Taking $G = -1$ and $m_i = q_i$, these results also describe a two-dimensional plasma with 1 charge $+e$ and 1 charge $-e$. According to equation (145), the partition function exists only for $T > T_*$ with

$$k_B T_* = \frac{e^2}{2}, \tag{157}$$

while the critical temperature appearing in the equation of state (148) is given by

$$k_B T_c = \frac{e^2}{4}. \tag{158}$$

Thus

$$T_* = 2T_c. \tag{159}$$

More general studies [36] show that the expression (157) of the collapse temperature remains valid for $N/2$ charges $+e$ and $N/2$ charges $-e$ where N is arbitrary. Indeed, the "catastrophic collapse" at $T = T_*$ corresponds to the formation of $N_* = N/2$ non-interacting pairs $(+,-)$ such as those studied here *individually*. It can therefore be obtained by studying the collapse of just *one* pair. Therefore, in 2D plasma physics, $T_* = 2T_c$ for any N (for a neutral plasma with two components). By comparison, in 2D gravity $T_* = \frac{N}{N-1}T_c$ (for a one component gas).

5.3 Relevance of a statistical description for N = 2 particles

We may wonder about the relevance of studying a system of $N = 2$ particles using statistical ensembles.

For Brownian particles, the N-body distribution function is governed by the Fokker-Planck equation (6) whose steady state is the canonical distribution (8). This is valid for *any* N. For $N = 1$, we recover the ordinary Brownian motion of a free particle submitted to a friction and a noise. The equilibrium state is the Maxwell-Boltzmann distribution predicted by statistical mechanics. For $N = 2$, we have two Brownian particles interacting via self-gravity and this system can be studied using the canonical ensemble. More generally, for Brownian particles, the canonical ensemble is justified for any N. For Hamiltonian systems, the relevance of a statistical description for a small number of particles is less clear. For example, $N = 2$ particles in gravitational interaction have a simple deterministic motion satisfying the Kepler laws. A statistical description may be justified, however, when the particles are hard spheres enclosed within a box so that their motion is very irregular due to collisions between themselves or against the walls (see the discussion in the review of Padmanabhan [4]). In that case, the microcanonical ensemble may be employed.

In any case, the density of states and the partition function can be defined mathematically for any N and it is interesting to compute these quantities exactly for $N = 2$ and compare these results with the ones obtained for $N \to +\infty$ by using a mean field approximation. For 3D self-gravitating systems, it is shown in [4,5] that the caloric curves (in microcanonical and canonical ensembles) obtained with $N = 2$ particles are already very close to those obtained for a large number of particles. Similarly, we have shown in the present paper that the same observation holds in $d = 2$ and $d = 1$ dimensions. This means that the mean field results are reliable even for a small number of particles.

6 Conclusion: analogies with chemotaxis and point vortex dynamics

In this paper, we have obtained the exact expression of the diffusion coefficient of a gas of self-gravitating Brownian particles in two dimensions. Two-dimensional gravity is a rare example where exact results can be obtained[6]. The diffusion coefficient exhibits a critical temperature T_c below which it becomes negative indicating finite time blow-up. In a finite domain, this critical temperature is slightly smaller than the collapse temperature T_* below which the partition function diverges. However, this difference is small, the ratio being of the order of $1 - 1/N$, and the two temperatures coincide at the thermodynamic (mean field) limit $N \to +\infty$. We may wonder what happens dynamically in the region $T_c < T < T_*$ since there is no equilibrium state in that case while we cannot prove finite time blow up from the Virial theorem. The system collapses for $T < T_*$ but it is possible that, for $T_c < T < T_*$, it takes an infinite time to form the Dirac peak. We stress that the computation of the diffusion coefficient brings only partial information on the dynamics of the self-gravitating Brownian gas. For example, it does not give any information concerning the precise evolution of the density distribution of the particles. To obtain the dynamical evolution of the density profile $\rho(\mathbf{r}, t)$, we need to solve the N-body equations (3, 4) or make a mean field approximation, valid in the proper thermodynamic limit $N \to +\infty$, where the problem reduces to the study of the Smoluchowski-Poisson (SP) system. Therefore, the complete description of the self-gravitating Brownian gas remains highly complicated and requires in general some approximations even if an exact result (1, 2) can be obtained.

The results obtained in this paper can also be relevant in mathematical biology [37]. They apply to a simplified version of the Keller-Segel model [38] modelling the chemotaxis of bacterial populations. The Keller-Segel model can be derived from a system of coupled Langevin equations of the form [39,40]:

$$\frac{d\mathbf{r}_\alpha}{dt} = \chi \nabla_\alpha c + \sqrt{2D}\mathbf{R}_\alpha(t), \tag{160}$$

$$\frac{\partial c}{\partial t} = -kc + D_c \Delta c + h \sum_{\alpha=1}^{N} \delta(\mathbf{r} - \mathbf{r}_\alpha(t)). \tag{161}$$

Here, $\mathbf{r}_\alpha(t)$ represents the position of a biological entity (cell, amoeba, bacteria,...) and $c(\mathbf{r}, t)$ is the concentration of the secreted chemical. The cells have a diffusion motion with diffusion coefficient D and they also move along a positive gradient of chemical (attractive chemotaxis $\chi > 0$). The chemical is produced by the cells with a strength h. It is in addition degraded at a rate k and diffuses with a diffusion coefficient D_c. The usual Keller-Segel model is recovered from these equations by making a mean field approximation leading to

$$\frac{\partial \rho}{\partial t} = D\Delta\rho - \chi \nabla \cdot (\rho \nabla c), \tag{162}$$

$$\frac{\partial c}{\partial t} = -kc + D_c \Delta c + h\rho. \tag{163}$$

Some authors have considered a simplified chemotactic model where the equation (161) for the chemical is replaced by a Poisson equation

$$\Delta c = -\lambda \sum_{\alpha=1}^{N} m\delta(\mathbf{r} - \mathbf{r}_\alpha(t)). \tag{164}$$

This can be justified in a limit of high diffusivity of the chemical and, in addition, for sufficiently dense systems [41,40]. In that case, the chemotactic problem becomes isomorphic to the study of self-gravitating Brownian particles described by equation (10), with a proper

[6] Of course, the results of this paper can be easily extended to power-law potentials of the form $u'(\xi) = G/\xi^{d-\alpha}$ in d dimensions. In that case, the critical dimension is $d_{crit} = \alpha + 1$.

re-interpretation of the parameters. In particular, the concentration $-c$ of the chemical plays the role of the gravitational potential Φ. We have the additional correspondences $\lambda \leftrightarrow S_d G$ and $\chi \leftrightarrow \mu m$. All the results of this paper can then be extended to the biological context provided that we make the appropriate substitutions. Furthermore, the consideration of the dimension $d = 2$ is particularly justified in biology since cellular organisms are often ascribed to move on a plane. In chemotaxis, we rather use the mass as a control parameter instead of the temperature. For example, the critical mass corresponding to the critical temperature T_c is

$$M_c = \frac{8\pi D}{\lambda\chi} + m, \tag{165}$$

and the collapse mass corresponding to the collapse temperature T_* is

$$M_* = \frac{8\pi D}{\lambda\chi}. \tag{166}$$

For a large number of cells ($M \gg m$) the two values coincide. For $M < M_*$, a box-confined system reaches an equilibrium state. For $M > M_*$, the system collapses and forms a Dirac peak corresponding to chemotactic aggregation. The adaptation of our results to the biological context is straightforward [42] so it will not be discussed in more detail here.

Let us finally consider the case of point vortices in two-dimensional hydrodynamics (see, e.g., [24]). The Hamiltonian of the point vortex gas is[7]

$$H = -\frac{1}{2\pi}\sum_{i<j}\gamma_i\gamma_j \ln|\mathbf{r}_i - \mathbf{r}_j|, \tag{167}$$

where the coordinates (x, y) are canonically conjugate [43]. The partition function in the canonical ensemble is given by

$$Z = \int e^{\frac{\beta}{2\pi}\sum_{i<j}\gamma_i\gamma_j \ln|\mathbf{r}_i-\mathbf{r}_j|}\prod_{k=1}^{N} d\mathbf{r}_k. \tag{168}$$

We note that there is no kinetic (quadratic) term in the Hamiltonian (167) in the usual sense, so that the temperature can take both positive or *negative* values [44]. By contrast, only positive temperatures are allowed in plasma physics and gravity. Let us first assume that the point vortices have the same circulation γ. At negative temperatures, the partition function is formally equivalent to that for self-gravitating systems. Therefore, at very negative inverse temperatures, point vortices have the tendency to "attract" each other and group themselves in a single aggregate (supervortex) of circulation $N\gamma$. This corresponds to a regime of high energies. This is similar to the formation of a Dirac peak in the gravitational problem for $T < T_*$. Thus, the partition function exists if and only if $\beta > \beta_*$ with

$$\beta_* = -\frac{8\pi}{N\gamma^2}. \tag{169}$$

This is the exact equivalent of the collapse temperature (118) as can be seen by taking $G = -1/(2\pi)$ and $m \leftrightarrow \gamma$. At positive temperatures, the partition function is formally equivalent to that for a non-neutral plasma of equal charges. Therefore, at large positive inverse temperatures, point vortices have the tendency to "repell" each other and accumulate on the boundary of the domain. We now consider N vortices with circulations $+\gamma$ and N vortices with circulations $-\gamma$. At positive temperatures, the partition function is formally equivalent to that for a neutral plasma. Therefore, at large positive inverse temperatures, the vortices of opposite sign have the tendency to "attract" each other and form N dipoles $(+,-)$. This corresponds to a regime of high negative energies. This is similar to the formation of N pairs $(+,-)$ in plasma physics for $T < T_*$. Thus, the partition function exists if and only if $\beta < \beta_*$ with

$$\beta_* = \frac{4\pi}{\gamma^2}. \tag{170}$$

This is the exact equivalent of the collapse temperature (121). At negative temperatures, the vortices of same sign have the tendency to "attract" each other and form an aggregate of positive circulation $N\gamma$ and an aggregate of negative circulation $-N\gamma$ (vortex dipole). This corresponds to a regime of high positive energies. This is similar to the formation of a Dirac peak in the gravitational problem, except that we have here *two* peaks: one with positive circulation and one with negative circulation. The partition function exists if and only if $\beta > \beta_*$ where β_* is given by equation (169) like in the case with one cluster of N particles. Therefore, the point vortex gas is very rich because it exhibits features similar to self-gravitating systems at negative temperatures and features similar to plasma systems at positive temperatures. We emphasize, however, that point vortices form a Hamiltonian system described by the microcanonical ensemble so that the control parameter is the energy (not the temperature) and the quantity of interest is the density of states (not the partition function). It can be shown that the density of states converges for any accessible energy. Therefore, the above collapse temperatures (169)-(170) represent lower and upper bounds on the caloric curve $\beta(E)$ which correspond to $E \to +\infty$ and $E \to -\infty$ respectively. The question to know what happens for $\beta < -\frac{8\pi}{N\gamma^2}$ or $\beta > \frac{4\pi}{\gamma^2}$ has *a priori* no sense in vortex dynamics since we cannot impose the temperature of the vortex gas.

[7] For brevity, we do not take into account the contribution of the images (in a bounded domain) because most of the following results are obtained by considering configurations where the vortices form compact clusters. They are therefore independent on boundary effects.

I acknowledge stimulating discussions with C. Sire and D. Dean on this subject. I am also grateful to D. Dallié for encouragements.

Appendix A: Alternative derivation of the kinetic equation

In this Appendix, we provide an alternative derivation of the exact kinetic equation equation (17). The Langevin equation (10) governing the time evolution of particle α in the strong friction limit can be written

$$\frac{d\mathbf{r}_\alpha}{dt} = -m_\alpha \mu_\alpha \nabla \Phi^{(\alpha)}(\mathbf{r}_\alpha) + \sqrt{2D_\alpha}\mathbf{R}_\alpha(t), \quad (171)$$

where

$$\Phi^{(\alpha)}(\mathbf{r},t) = \sum_{\beta\neq\alpha} m_\beta u(|\mathbf{r}-\mathbf{r}_\beta(t)|), \quad (172)$$

denotes the exact potential created by the other particles. We define the one and two-body probability distributions by

$$P_\alpha(\mathbf{r},t) = \langle \delta(\mathbf{r}-\mathbf{r}_\alpha(t))\rangle, \quad (173)$$

$$P_{\alpha,\beta}(\mathbf{r},\mathbf{r}',t) = \langle \delta(\mathbf{r}-\mathbf{r}_\alpha(t))\delta(\mathbf{r}'-\mathbf{r}_\beta(t))\rangle. \quad (174)$$

Differentiating equation (173) with respect to time, we obtain

$$\frac{\partial P_\alpha}{\partial t} = -\nabla\cdot\langle \dot{\mathbf{r}}_\alpha(t)\delta(\mathbf{r}-\mathbf{r}_\alpha(t))\rangle. \quad (175)$$

Substituting from equation (171) in equation (175), we find that

$$\frac{\partial P_\alpha}{\partial t} = -\nabla\cdot\langle\sqrt{2D_\alpha}\mathbf{R}_\alpha(t)\delta(\mathbf{r}-\mathbf{r}_\alpha(t))\rangle + \nabla\cdot\langle m_\alpha\mu_\alpha\nabla\Phi^{(\alpha)}(\mathbf{r}_\alpha(t))\delta(\mathbf{r}-\mathbf{r}_\alpha(t))\rangle. \quad (176)$$

The first term is the standard term that appears in deriving the Fokker-Planck equation for a pure random walk; it leads to a term proportional to the Laplacian of P_α:

$$-\nabla\cdot\langle\sqrt{2D_\alpha}\mathbf{R}_\alpha(t)\delta(\mathbf{r}-\mathbf{r}_\alpha(t))\rangle = D_\alpha\Delta P_\alpha. \quad (177)$$

To evaluate the second term, we first write

$$\Phi^{(\alpha)}(\mathbf{r},t) = \int d\mathbf{r}' \sum_{\beta\neq\alpha} m_\beta u(|\mathbf{r}-\mathbf{r}'|)\delta(\mathbf{r}'-\mathbf{r}_\beta(t)). \quad (178)$$

This yields

$$\langle\nabla\Phi^{(\alpha)}(\mathbf{r}_\alpha(t))\delta(\mathbf{r}-\mathbf{r}_\alpha(t))\rangle = \int d\mathbf{r}'\sum_{\beta\neq\alpha} m_\beta\nabla u(|\mathbf{r}-\mathbf{r}'|)\langle\delta(\mathbf{r}-\mathbf{r}_\alpha(t))\delta(\mathbf{r}'-\mathbf{r}_\beta(t))\rangle = \int d\mathbf{r}'\sum_{\beta\neq\alpha} m_\beta P_{\alpha,\beta}(\mathbf{r},\mathbf{r}',t)\nabla u(|\mathbf{r}-\mathbf{r}'|). \quad (179)$$

Substituting equations (177) and (179) in equation (176), we get the exact kinetic equation (17).

We can easily extend this approach so as to take into account the inertia of the particles. In that case, the dynamical evolution of particle α is described by stochastic equations of the form

$$\frac{d\mathbf{r}_\alpha}{dt} = \mathbf{v}_\alpha, \quad (180)$$

$$\frac{d\mathbf{v}_\alpha}{dt} = -\xi_\alpha\mathbf{v}_\alpha - \nabla\Phi^{(\alpha)}(\mathbf{r}_\alpha(t)) + \sqrt{2D'_\alpha}\mathbf{R}_\alpha(t). \quad (181)$$

We define the one-body probability distributions in phase space by

$$P_\alpha(\mathbf{r},\mathbf{v},t) = \langle\delta(\mathbf{r}-\mathbf{r}_\alpha(t))\delta(\mathbf{v}-\mathbf{v}_\alpha(t))\rangle, \quad (182)$$

where the brackets denote an average over the noise. Similarly, the two-body probability distribution in phase space is

$$P_{\alpha,\beta}(\mathbf{r},\mathbf{v};\mathbf{r}',\mathbf{v}',t) = \langle\delta(\mathbf{r}-\mathbf{r}_\alpha(t))\delta(\mathbf{v}-\mathbf{v}_\alpha(t)) \times\delta(\mathbf{r}'-\mathbf{r}_\beta(t))\delta(\mathbf{v}'-\mathbf{v}_\beta(t))\rangle. \quad (183)$$

Taking the time derivative of P_α, we get

$$\frac{\partial P_\alpha}{\partial t} = -\frac{\partial}{\partial\mathbf{r}}\cdot\langle\dot{\mathbf{r}}_\alpha(t)\delta(\mathbf{r}-\mathbf{r}_\alpha(t))\delta(\mathbf{v}-\mathbf{v}_\alpha(t))\rangle - \frac{\partial}{\partial\mathbf{v}}\cdot\langle\dot{\mathbf{v}}_\alpha(t)\delta(\mathbf{r}-\mathbf{r}_\alpha(t))\delta(\mathbf{v}-\mathbf{v}_\alpha(t))\rangle. \quad (184)$$

Inserting the equations of motion (180)-(181) in equation (184), we obtain

$$\frac{\partial P_\alpha}{\partial t} = -\frac{\partial}{\partial\mathbf{r}}\cdot\langle\mathbf{v}_\alpha(t)\ \delta(\mathbf{r}-\mathbf{r}_\alpha(t))\delta(\mathbf{v}-\mathbf{v}_\alpha(t))\rangle + \frac{\partial}{\partial\mathbf{v}}\cdot\langle\xi_\alpha\mathbf{v}_\alpha(t)\ \delta(\mathbf{r}-\mathbf{r}_\alpha(t))\delta(\mathbf{v}-\mathbf{v}_\alpha(t))\rangle + \frac{\partial}{\partial\mathbf{v}}\cdot\langle\nabla\Phi^{(\alpha)}(\mathbf{r}_\alpha(t))\ \delta(\mathbf{r}-\mathbf{r}_\alpha(t))\delta(\mathbf{v}-\mathbf{v}_\alpha(t))\rangle - \frac{\partial}{\partial\mathbf{v}}\cdot\langle\sqrt{2D'_\alpha}\mathbf{R}_\alpha(t)\ \delta(\mathbf{r}-\mathbf{r}_\alpha(t))\delta(\mathbf{v}-\mathbf{v}_\alpha(t))\rangle. \quad (185)$$

The first two terms are straightforward to evaluate. The fourth term is the standard term that appears in deriving the Fokker-Planck equation for a pure random walk; it leads to a term proportional to the Laplacian of P_α in velocity space. The third term can be evaluated by inserting the expression (178) in equation (185). Finally, we obtain the exact equation

$$\frac{\partial P_\alpha}{\partial t} + \mathbf{v}\cdot\frac{\partial P_\alpha}{\partial\mathbf{r}} - \frac{\partial}{\partial\mathbf{v}}\cdot\int d\mathbf{r}'\sum_{\beta\neq\alpha} m_\beta\nabla u(|\mathbf{r}-\mathbf{r}'|) \times P_{\alpha,\beta}(\mathbf{r},\mathbf{v};\mathbf{r}',t) = \frac{\partial}{\partial\mathbf{v}}\cdot\left[D'_\alpha\frac{\partial P_\alpha}{\partial\mathbf{v}} + \xi_\alpha P_\alpha\mathbf{v}\right], \quad (186)$$

where the statistical correlations are encapsulated in the two-body probability distribution

$$P_{\alpha,\beta}(\mathbf{r},\mathbf{v};\mathbf{r}',t) = \langle\delta(\mathbf{r}-\mathbf{r}_\alpha(t))\delta(\mathbf{v}-\mathbf{v}_\alpha(t))\delta(\mathbf{r}'-\mathbf{r}_\beta(t))\rangle. \quad (187)$$

If we implement a mean field approximation

$$P_{\alpha,\beta}(\mathbf{r},\mathbf{v};\mathbf{r}',t)=P_\alpha(\mathbf{r},\mathbf{v},t)P_\beta(\mathbf{r}',t), \quad (188)$$

the preceding equation can be rewritten

$$\frac{\partial P_\alpha}{\partial t}+\mathbf{v}\cdot\frac{\partial P_\alpha}{\partial \mathbf{r}}-\frac{\partial P_\alpha}{\partial \mathbf{v}}\cdot\nabla\int d\mathbf{r}'\sum_{\beta\neq\alpha}m_\beta u(|\mathbf{r}-\mathbf{r}'|)P_\beta(\mathbf{r}',t)= \frac{\partial}{\partial\mathbf{v}}\cdot\left[D'_\alpha\frac{\partial P_\alpha}{\partial\mathbf{v}}+\xi_\alpha P_\alpha\mathbf{v}\right]. \quad (189)$$

If we extend the sum over all β and introduce the spatial density (63), we obtain

$$\frac{\partial P_\alpha}{\partial t}+\mathbf{v}\cdot\frac{\partial P_\alpha}{\partial \mathbf{r}}-\frac{\partial P_\alpha}{\partial \mathbf{v}}\cdot\nabla\int d\mathbf{r}' u(|\mathbf{r}-\mathbf{r}'|)\rho(\mathbf{r}',t)= \frac{\partial}{\partial\mathbf{v}}\cdot\left[D'_\alpha\frac{\partial P_\alpha}{\partial\mathbf{v}}+\xi_\alpha P_\alpha\mathbf{v}\right]. \quad (190)$$

This can be rewritten in the form of a mean field Kramers equation

$$\frac{\partial P_\alpha}{\partial t}+\mathbf{v}\cdot\frac{\partial P_\alpha}{\partial \mathbf{r}}-\nabla\Phi\cdot\frac{\partial P_\alpha}{\partial \mathbf{v}}=\frac{\partial}{\partial\mathbf{v}}\cdot\left[D'_\alpha\frac{\partial P_\alpha}{\partial\mathbf{v}}+\xi_\alpha P_\alpha\mathbf{v}\right], \quad (191)$$

where

$$\Phi(\mathbf{r},t)=\int\rho(\mathbf{r}',t)u(|\mathbf{r}-\mathbf{r}'|)d\mathbf{r}', \quad (192)$$

is the potential.

Appendix B: The case T = 0

Here, we assume that all the particles have the same mass m. At $T=0$ (no noise), the equations of motion (10) in the overdamped limit become

$$\frac{d\mathbf{r}_\alpha}{dt}=-\frac{1}{\xi}\nabla\Phi_{ex}(\mathbf{r}_\alpha,t), \quad (193)$$

where $\Phi_{ex}(\mathbf{r},t)$ is the exact gravitational potential that is solution of the Poisson equation

$$\Delta\Phi_{ex}=S_dG\rho_{ex}, \quad (194)$$

with the exact density field

$$\rho_{ex}(\mathbf{r},t)=\sum_{\alpha=1}^N m\delta(\mathbf{r}-\mathbf{r}_\alpha(t)). \quad (195)$$

Taking the time derivative of equation (195), we get

$$\frac{\partial\rho_{ex}}{\partial t}=-\sum_{\alpha=1}^N m\nabla\cdot(\dot{\mathbf{r}}_\alpha\delta(\mathbf{r}-\mathbf{r}_\alpha(t))). \quad (196)$$

Substituting equations (193) in equation (196), we obtain

$$\frac{\partial\rho_{ex}}{\partial t}=\frac{1}{\xi}\sum_{\alpha=1}^N m\nabla\cdot(\nabla\Phi_{ex}(\mathbf{r}_\alpha(t),t)\delta(\mathbf{r}-\mathbf{r}_\alpha(t))) =\frac{1}{\xi}\nabla\cdot\left(\nabla\Phi_{ex}(\mathbf{r},t)\sum_{\alpha=1}^N m\delta(\mathbf{r}-\mathbf{r}_\alpha(t))\right), \quad (197)$$

so that, finally,

$$\frac{\partial\rho_{ex}}{\partial t}=\frac{1}{\xi}\nabla\cdot(\rho_{ex}\nabla\Phi_{ex}(\mathbf{r},t)). \quad (198)$$

This exact equation, where ρ_{ex} is expressed in terms of Dirac functions and Φ_{ex} is given by equation (194), contains exactly the same information as the deterministic equations (193).

For the inertial model, at $T=0$, the equations of motion are

$$\frac{d\mathbf{r}_\alpha}{dt}=\mathbf{v}_\alpha, \quad (199)$$

$$\frac{d\mathbf{v}_\alpha}{dt}=-\xi\mathbf{v}_\alpha-\nabla\Phi_{ex}(\mathbf{r}_\alpha,t). \quad (200)$$

Introducing the exact distribution function

$$f_{ex}(\mathbf{r},\mathbf{v},t)=\sum_{\alpha=1}^N m\delta(\mathbf{r}-\mathbf{r}_\alpha(t))\delta(\mathbf{v}-\mathbf{v}_\alpha(t)), \quad (201)$$

and taking the time derivative of equation (201), we get

$$\frac{\partial f_{ex}}{\partial t}=-\sum_{\alpha=1}^N m\nabla\cdot(\dot{\mathbf{r}}_\alpha\delta(\mathbf{r}-\mathbf{r}_\alpha(t))\delta(\mathbf{v}-\mathbf{v}_\alpha(t))) -\sum_{\alpha=1}^N m\frac{\partial}{\partial\mathbf{v}}\cdot(\dot{\mathbf{v}}_\alpha\delta(\mathbf{r}-\mathbf{r}_\alpha(t))\delta(\mathbf{v}-\mathbf{v}_\alpha(t))). \quad (202)$$

Substituting equations (199, 200) in equation (202) and following a procedure similar to that developed previously, we obtain

$$\frac{\partial f_{ex}}{\partial t}+\mathbf{v}\cdot\frac{\partial f_{ex}}{\partial\mathbf{r}}-\nabla\Phi_{ex}\cdot\frac{\partial f_{ex}}{\partial\mathbf{v}}=\xi\frac{\partial}{\partial\mathbf{v}}(f_{ex}\mathbf{v}). \quad (203)$$

This exact equation, where f_{ex} is expressed in terms of Dirac functions, contains exactly the same information as the deterministic equations (199, 200). For $\xi=0$, it reduces to the Klimontovich equation of plasma physics.

Appendix C: The case N = 2 in one dimension

In this Appendix, we study the statistical mechanics of two particles in gravitational interaction in $d=1$ where explicit results can be obtained.

C.1 The microcanonical ensemble

In $d = 1$, the Hamiltonian of two particles in gravitational interaction can be written

$$H = \frac{1}{2}MV^2 + \frac{1}{2}\mu v^2 + Gm_1m_2|r|, \tag{204}$$

where we have used the coordinates (R, V) of the center of mass and the coordinates (r, v) of the reduced particle. We note that the energy takes only positive values. Introducing $x_1 = (M/2)^{1/2}V$ and $x_2 = (\mu/2)^{1/2}v$, the surface of phase space with energy less than E can be rewritten

$$\Gamma(E) = \frac{8R}{\sqrt{M\mu}} \int_0^{r_m} dr \int_{\|\mathbf{x}\| \leq \sqrt{E - Gm_1m_2r}} d^2x. \tag{205}$$

The last integral represents the surface of a disc with radius $\sqrt{E - Gm_1m_2r}$. Thus, we get

$$\Gamma(E) = \frac{8\pi R}{\sqrt{M\mu}} \int_0^{r_m} (E - Gm_1m_2r)dr. \tag{206}$$

The range of integration r_m is such that $E - Gm_1m_2r \geq 0$ and $r \leq R$. Therefore, $r_m = E/Gm_1m_2$ if $E \leq Gm_1m_2R$ and $r_m = R$ if $E \geq Gm_1m_2R$. The density of states in the microcanonical ensemble is $g(E) = d\Gamma/dE$. Introducing

$$A = \frac{8\pi}{\sqrt{M\mu}}, \quad \epsilon = \frac{E}{Gm_1m_2R}, \quad t = \frac{k_BT}{Gm_1m_2R}, \tag{207}$$

we get

$$g(\epsilon) = ARr_m, \tag{208}$$

with $r_m = \epsilon R$ if $0 \leq \epsilon \leq 1$ and $r_m = R$ if $\epsilon \geq 1$.

The caloric curve $t = t(\epsilon)$ and the equation of state in the microcanonical ensemble can be obtained from equations (133, 135) and the exact expression (208) of the density of states. For $0 \leq \epsilon \leq 1$, we obtain

$$g(\epsilon) = AR^2\epsilon, \tag{209}$$

$$k_BT = E \quad \rightarrow t = \epsilon, \tag{210}$$

$$PV = 2k_BT. \tag{211}$$

For $\epsilon \geq 1$, we obtain

$$g(\epsilon) = AR^2, \tag{212}$$

$$k_BT = +\infty \quad \rightarrow t = +\infty, \tag{213}$$

$$PV = 2k_BT = +\infty. \tag{214}$$

The caloric curve in the microcanonical ensemble is represented in Figure 4.

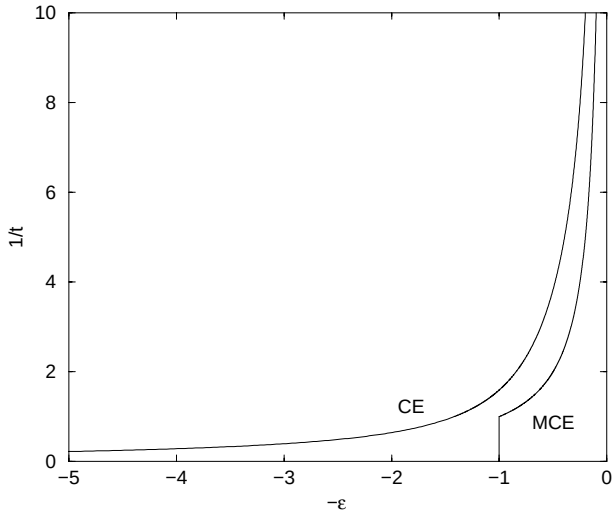

Fig. 4. Caloric curve in microcanonical (MCE) and canonical (CE) ensembles for the binary star model $N = 2$ in one dimension. The caloric curves are similar in the two ensembles (without negative specific heat region) so that there is no phase transition contrary to the situation in $d = 3$ [4,5]. There exists equilibrium states for all accessible energies $E \geq 0$ in MCE and for all temperatures in CE. It can be of interest to compare these curves with the caloric curve obtained within the mean field approximation valid for $N \gg 1$ which presents a similar behaviour [12].

C.2 The canonical ensemble

For $N = 2$ particles in gravitational interaction, the partition function in the canonical ensemble can be written

$$Z = \int e^{-\beta M\frac{V^2}{2}} e^{-\beta\mu\frac{v^2}{2}} e^{-\beta m_1m_2G|r|} dRdV drdv. \tag{215}$$

Integrating over the velocities and over the position of the center of mass[8], we get

$$Z = \frac{8\pi R}{\beta\sqrt{M\mu}} \int_0^R e^{-\beta Gm_1m_2r}\, dr. \tag{216}$$

Therefore, the partition function is given by

$$Z = \frac{8\pi R}{\beta^2 G(m_1m_2)^{3/2}} \left(1 - e^{-\beta Gm_1m_2R}\right). \tag{217}$$

The pressure can be computed from equation (82) leading to

$$PV = k_BT \left[1 + \frac{Gm_1m_2\beta R}{e^{\beta Gm_1m_2R} - 1}\right]. \tag{218}$$

For $T \to +\infty$, we have $PV = 2k_BT$ and for $T \to 0$, we have $PV = k_BT$ (this is the pressure created by an effective *single* particle resulting from the collapse of the

[8] In fact, the domain of integration of r depends on R. Therefore, our treatment (see also Sect. 5) is approximate and will be improved in a future work.

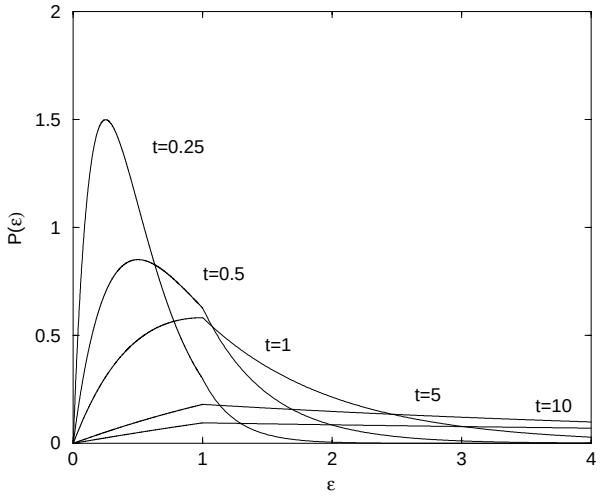

Fig. 5. Distribution of energies for the binary star model $N = 2$ in one dimension in the canonical ensemble. We have plotted this distribution for different values of the temperature.

two particles). The average energy obtained from equations (151) and (217) is

$$\langle \epsilon \rangle = 2t - \frac{1}{e^{1/t} - 1}. \qquad (219)$$

It behaves like $\langle \epsilon \rangle \sim 2t$ for $t \to 0$ and like $\langle \epsilon \rangle \sim t$ for $t \to +\infty$. The caloric curve in the canonical ensemble is represented in Figure 4 and it is compared with the microcanonical caloric curve. Finally, the energy distribution at temperature T in the canonical ensemble obtained from equations (153, 217) and (208) is given by

$$P(\epsilon) = \frac{\epsilon e^{-\epsilon/t}}{t^2(1 - e^{-1/t})}, \qquad (0 \leq \epsilon \leq 1), \qquad (220)$$

$$P(\epsilon) = \frac{e^{-\epsilon/t}}{t^2(1 - e^{-1/t})} \qquad (\epsilon \geq 1). \qquad (221)$$

The distribution of energies is represented in Figure 5 for different temperatures.

References

1. *Dynamics and Thermodynamics of Systems with Long Range Interactions*, edited by T. Dauxois, S. Ruffo, E. Arimondo, M. Wilkens, Lect. Not. in Phys. **602** (Springer, 2002)
2. P.H. Chavanis, Physica A **361**, 55 (2006)
3. P.H. Chavanis, Physica A **361**, 81 (2006)
4. T. Padmanabhan, Phys. Rep. **188**, 285 (1990)
5. P.H. Chavanis, Int. J. Mod. Phys. B **20**, 3113 (2006)
6. D. Lynden-Bell, R. Wood, Mon. Not. R. Astron. Soc. **138**, 495 (1968)
7. P.H. Chavanis, A&A **381**, 340 (2002)
8. J. Binney, S. Tremaine, *Galactic Dynamics* (Princeton Series in Astrophysics, 1987)
9. D. Heggie, P. Hut, *The Gravitational Million-Body Problem: A Multidisciplinary Approach to Star Cluster Dynamics* (Cambridge University Press, 2003)
10. W.C. Saslaw, *Gravitational Physics of Stellar and Galactic Systems* (Cambridge University Press, 1985)
11. P.H. Chavanis, C. Rosier, C. Sire, Phys. Rev. E **66**, 036105 (2002)
12. C. Sire, P.H. Chavanis, Phys. Rev. E **66**, 046133 (2002)
13. P.H. Chavanis, C. Sire, Phys. Rev. E **69**, 016116 (2004)
14. C. Sire, P.H. Chavanis, Phys. Rev. E **69**, 066109 (2004)
15. P.H. Chavanis, C. Sire, Phys. Rev. E **70**, 026115 (2004)
16. C. Sire, P.H. Chavanis, Banach Center Publ. **66**, 287 (2004)
17. P.H. Chavanis, M. Ribot, C. Rosier, C. Sire, Banach Center Publ. **66**, 103 (2004)
18. J. Sopik, C. Sire, P.H. Chavanis, Phys. Rev. E **72**, 026105 (2005)
19. P.H. Chavanis, C. Sire, Phys. Rev. E **73**, 066103 (2006)
20. P.H. Chavanis, C. Sire, Phys. Rev. E **73**, 066104 (2006)
21. P.H. Chavanis, C. R. Physique, **7**, 331 (2006)
22. P.H. Chavanis, A&A **356**, 1089 (2000)
23. P.H. Chavanis, Eur. Phys. J. B **52**, 433 (2006)
24. P.H. Chavanis, *Statistical Mechanics of Two-dimensional Vortices and Stellar Systems*, in Dynamics and Thermodynamics of Systems with Long Range Interactions, edited by T. Dauxois, S. Ruffo, E. Arimondo, M. Wilkens, Lect. Not. in Phys. **602** (Springer, 2002), e-print `arXiv:cond-mat/0212223`
25. P.H. Chavanis, Eur. Phys. J. B **54**, 525 (2006)
26. J. Sopik, C. Sire, P.H. Chavanis, Phys. Rev. E **74**, 011112 (2006)
27. W. Götze, *Liquids, Freezing amd the Glass Transition* (Les Houches, North-Holland, Amsterdam, 1989)
28. W.B. Russel, D.A. Saville, W.R. Schowalter, *Colloidal Dispersions* (Cambridge University Press, Cambridge, England, 1989)
29. D.S. Dean, A. Lefèvre, Phys. Rev. E **69**, 061111 (2004)
30. A. Einstein, *Investigations on the Theory of Brownian Movement*, edited by R. Fürth, translated by A.D. Cowper (Dover, New York, 1956)
31. A.M. Salzberg, S. Prager, J. Chem. Phys. Lett. **38**, 2587 (1964)
32. A.M. Salzberg, J. Math. Phys. **6**, 158 (1965)
33. J. Katz, D. Lynden-Bell, Mon. Not. R. Astron. Soc. **184**, 709 (1978)
34. M. Kiessling, Comm. Pure Appl. Math. **46**, 27 (1993)
35. E. Caglioti, P.L. Lions, C. Marchioro, M. Pulvirenti, Commun. Math. Phys. **143**, 501 (1992)
36. A. Alastuey, Ann. Phys. Fr. **11**, 653 (1986)
37. J.D. Murray, *Mathematical Biology* (Springer, Berlin, 1991)
38. E.F. Keller, L.A. Segel, J. Theor. Biol. **30**, 225 (1971)
39. T.J. Newman, R. Grima, Phys. Rev. E **70**, 051916 (2004)
40. P.H. Chavanis, C. Sire, Physica A (in press), `arXiv:0706.3974`
41. W. Jäger, S. Luckhaus, Trans. Amer. Math. Soc. **329**, 819 (1992)
42. P.H. Chavanis, Physica A (in press), `arXiv:0706.3603`
43. G. Kirchhoff, in *Lectures in Mathematical Physics, Mechanics* (Teubner, Leipzig, 1877)
44. L. Onsager, Nuovo Cimento, Suppl. **6**, 279 (1949)